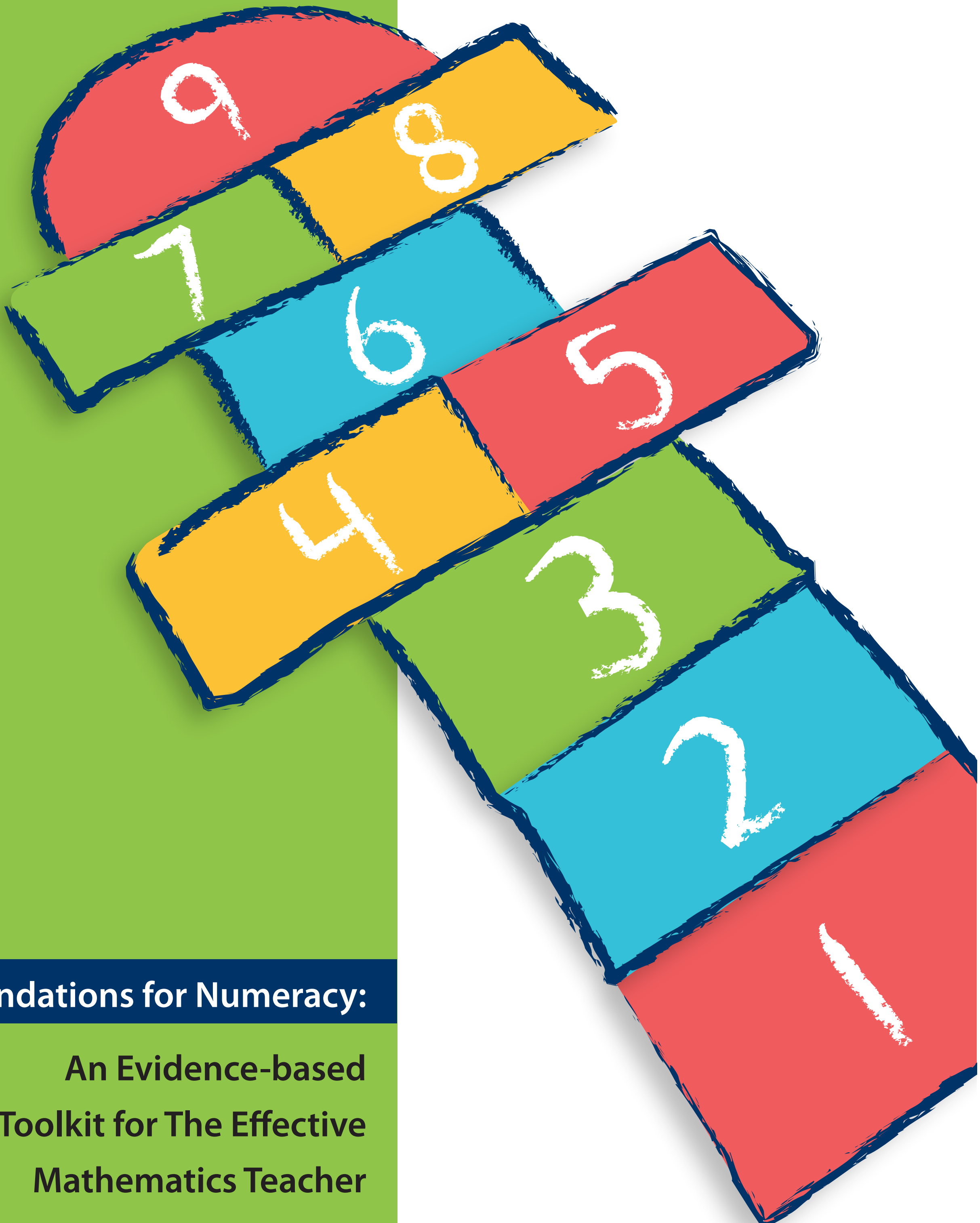

Foundations for Numeracy:

An Evidence-based Toolkit for The Effective Mathematics Teacher

Réseau canadien de recherche sur le langage et l'alphabétisation
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FÉDÉRATION CANADIENNE DES SERVICES DE GARDE À L'ENFANCE

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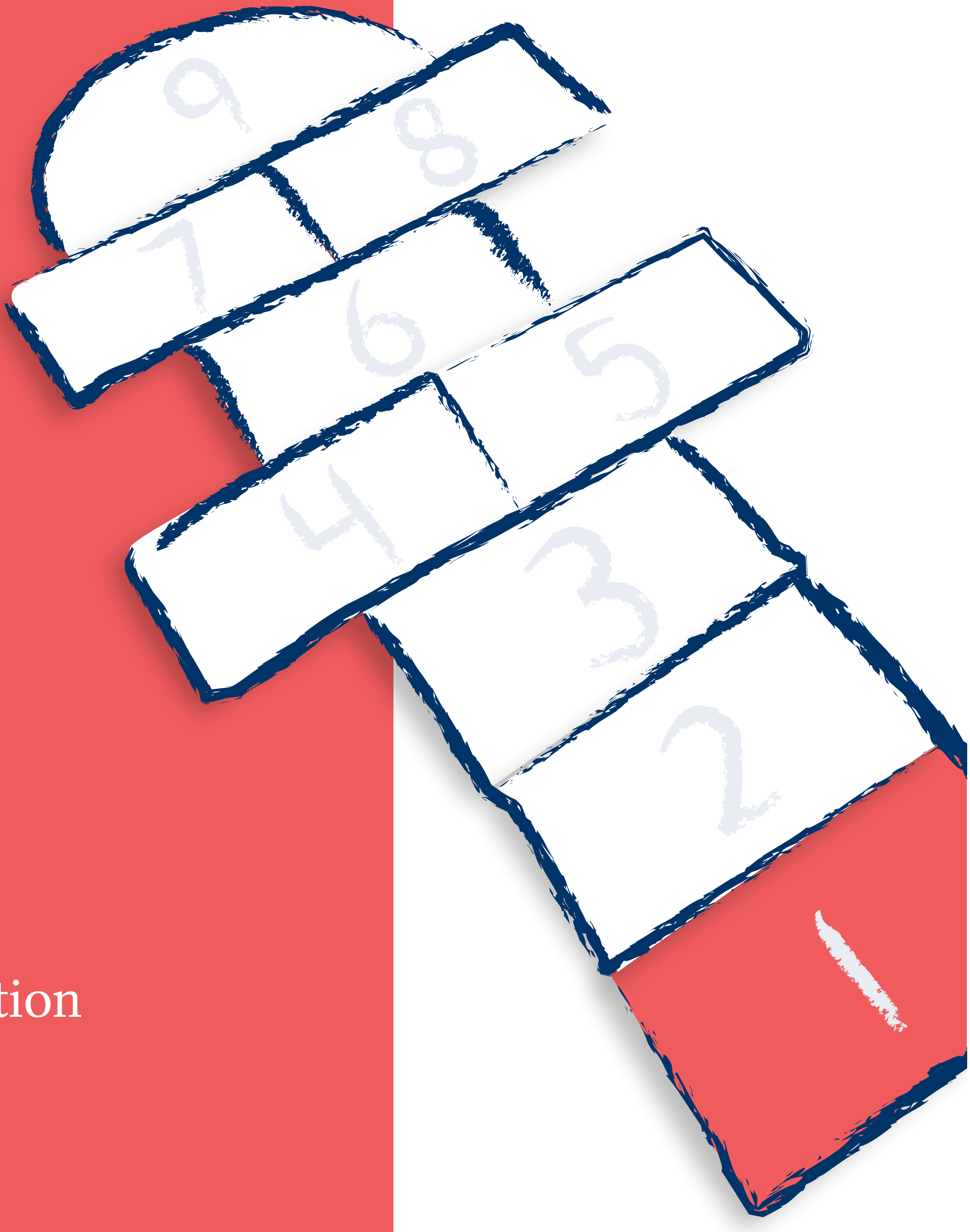
Introduction



INTRODUCTION

This resource kit was created for early learning practitioners and teachers to help support the development of numeracy skills of children in their care. The information presented in the kit is based on a comprehensive review of recent well-designed research studies on the learning and teaching of mathematics. The findings of these studies are communicated in an accessible format, making this resource an effective reference tool that can be used in daily practice.

The kit is divided into two volumes: one for early learning and child care practitioners and the other for elementary school teachers. Each volume includes a research summary and several additional components. The current volume, which is intended for teachers who work with children from Kindergarten to Grade 6, includes the following components:

- Creating a Math-Rich Environment (tips for elementary school teachers to make their classrooms more inviting and conducive to mathematics learning)
- The Development of Mathematics in the Elementary Years (a research summary for teachers)
- Resources for Elementary School Teachers (a list of both print and online resources on supporting numeracy)
- Activities (learning activities for children from Kindergarten to Grade 6)
- Glossary of Terms (definitions of technical terms related to numeracy)
- Numeracy Poster (a poster in the form of a number line)

The kit is intended to supplement and enhance elementary school teachers' prior knowledge of mathematics teaching, as well as to introduce new information on the teaching of mathematical concepts. It allows teachers to stay up-to-date on the latest advances in mathematics teaching and learning, and helps teachers to identify the most effective approaches that can be used in classroom environments. It is a useful learning resource for teachers in training and a practical professional development resource for those who are already teaching.

Numeracy during the elementary school years

The elementary school years are a time of growth and new, exciting experiences for children. Children undergo many developmental changes between the ages of six and twelve, particularly in terms of their cognitive development. Such changes, by extension, influence their numeracy development. During the elementary school years, children acquire the ability to process and retain new information, and to solve increasingly complex mathematical problems. Consequently, elementary school children are able to progress from the rudiments of symbolic number system knowledge to basic algebra.

Children's mathematical development is closely tied to the opportunities that they experience in elementary school. Accordingly, creating a math-rich classroom environment and employing a variety of teaching approaches and learning activities can be extraordinarily beneficial to students. Recent thinking in math education has shifted from an emphasis on the teaching and practicing of algorithms (e.g., memorizing arithmetic facts or completing pages of math problems) to focusing on reasoning and problem solving and subsequent application to real world problems.

It is imperative for teachers to remain cognizant of the fact that a child's acquisition of mathematical knowledge will vary according to individual experience, disposition, and even brain maturation. Yet as a group, and over a reasonable period of time, acquired knowledge and proficiency with mathematics will improve for students, given a dynamic instructor who employs developmentally appropriate strategies. Dedicated teachers who provide support for their students' mathematics development in the elementary school years will pave the way for each child to realize his or her true mathematical potential.

2

The Research

RESEARCH SUMMARY

To effectively support children's learning, educators need information based on existing research evidence. They can then integrate this knowledge with their professional experience and their understanding of children's needs. This research summary draws on a variety of sources related to the learning and teaching of mathematics and summarizes their findings. Educators will find information here about current thinking on the principles that underlie learning and development, particularly as they relate to mathematics.

In Part 1, we will focus on the cognitive processes that influence mathematics learning and achievement. We begin with a discussion of what the research tells us about three levels of cognition: information processing, mental representations, and metacognitive processes (thinking about thinking). Following this, we discuss research findings on the social and emotional factors that influence learning, in particular children's learning goals, motivation, beliefs about learning, and the influence of math anxiety on achievement.

In Part 2, the focus shifts to the development of mathematical concepts from the early years (preschool) through the transition to school (Kindergarten) and into the elementary years (Grades 1 to 6). From a child's early mathematical abilities and skills to the more formalized sets of rules and strategies learned in school, this section will focus on some of the key underlying concepts and widely applicable skills. These include numerosity, cardinality, ordinality, problem solving, the mental number line, fractions, estimation, arithmetic, and proportional reasoning.

PART 1: THE PROCESSES UNDERLYING CHILDREN'S LEARNING

What can educators learn from cognitive science that they can apply to their learning environments and classrooms? Cognitive scientists study every type of human learning and can provide insight into many of the underlying processes that guide children's learning:

- **information processing** (e.g., attention, working memory, and the retrieval, transfer, and retention of information);
- **mental representations** (e.g., conceptual and procedural knowledge); and
- **metacognitive processes** (e.g., processes that control mental operations, such as strategy selection and self-monitoring behaviours).

These processes can be considered as the "cognitive building blocks" of children's achievement. We will discuss each in turn in order to better understand how children learn and how educators can best support their learning (National Mathematics Advisory Panel [NMAP], 2008).

Cognitive factors are not the only ones that contribute to children's achievement in mathematics. A child's motivation, capacity for self-regulation, and anxiety about math can all have a strong effect on cognitive processing, and thus on achievement. Good teaching takes all of these factors into account, recognizing that social and emotional factors, as well as children's goals and beliefs about learning, are critical components of the learning process.

COGNITIVE PROCESSES

Information Processing

The first step in many types of learning or processing of information is to **focus our attention**. However, as we are well aware, there is a limit to how many things we can pay attention to at once. Attention changes with age: under many conditions younger children are less attentive than adults, and thus more prone to distractions (Cowan, Elliott, & Saults, 2002). However, our ability to attend to information is not entirely out of our control; it can also be improved with practice (Baumeister, 2005; Gailliot, Plant, Butz, & Baumeister, 2007; Muraven, Baumeister, & Tice, 1999).

Once we focus our attention on information, it is encoded in our **working memory**.

Working memory refers to the ability to keep information active in your mind while you use that information to perform an operation. For instance, if we ask a child to solve the problem "3 plus 5" without writing anything down, she must keep the information "3 plus 5" active, decode the meaning of the individual symbols, and then carry out a number of operations to reach the answer. In a way, the function of working memory in problem solving is like learning how to drive a standard transmission car. For a new driver, focussing on traffic and shifting is very demanding, and – just as with math processing – sometimes there are accidents, or errors.

In this driving analogy, working memory can be described as the "attention-driven control of information" (Baddeley, 1986, 2000; Engle, Conway, Tuholski, & Shisler, 1995). Depending on the type of incoming information, working memory stores information in one of three systems: the language-based phonetic buffer (e.g., remembering a phone number), the visuospatial sketch pad (e.g., remembering a visual pattern), or the episodic buffer (where information from long-term memory and the world is combined). Working memory has been strongly associated with academic learning, including in mathematics. A deficient working memory is one source of learning problems encountered by children with learning disabilities in mathematics. Conversely, a strong working memory is a major factor behind the accelerated learning shown by gifted children (NMAP, 2008, p. 4-5).

Attention and working memory ability increase with age and there are ways to improve children's working memory at any age. The most effective way to improve working memory is to help children achieve the quick, easy, and effortless retrieval of information from long-term memory, particularly of basic skills, facts, and procedures (Schneider & Shiffrin, 1977; Shiffrin & Schneider, 1977).

This quick retrieval, called "automaticity," is only achieved through practice (e.g., Cooper & Sweller, 1987). For most types of learning, automaticity of basic skills frees up working memory for more complex aspects of problem solving, such as creating mental pictures of the information, analyzing the problem,

choosing and employing a strategy, and checking the answer obtained (NMAP, 2008). Again, we can make a comparison to learning to drive a standard transmission car. With practice, skills like changing gears and checking the blind spot are mastered. When these once demanding tasks become automatic, the driver's attention can be directed to the road ahead.

MENTAL REPRESENTATIONS

Types of Knowledge

There are three types of knowledge relevant to math:

- **factual knowledge** is information that can be learned by memorization and repetition (i.e., rote learning), such as knowing that 2 + 2 = 4. It also refers to memory of specific events and information.
- **conceptual knowledge** is the knowledge of why and how a procedure works, and includes general knowledge and understanding of a subject (Hiebert & Lefevre, 1986). It is information stored in long-term memory, acquired through thoughtful reflection over a long period.
  For example, knowing that when we count, the last number we say represents how many items are in the set.
- **procedural knowledge** describes the implicit memory for cognitive and motor sequences and skills, in short, knowing how to complete an activity or a task. For example, knowing how to solve the problem 2 + 3 by continuing to count "3, 4, 5…" (Hunt & Ellis, 1994; NMAP, 2008).

These three types of knowledge mutually support each other to facilitate learning and understanding (NMAP, 2008). Conceptual and procedural knowledge in particular have been shown to be positively correlated: when one increases, so does the other (Rittle-Johnson & Siegler, 1998). For instance, research has shown that an early measure of the degree to which elementary students have conceptual understanding predicts not only their procedural ability in the same unit, but also procedural skill in the future (Hiebert & Wearne, 1996). Conceptual, procedural, and factual knowledge are all important for success in mathematics: "conceptual understanding of mathematical operations, fluent execution of procedures, and fast access to number combinations together support effective and efficient problem solving" (NMAP, 2008, p. 26).

Although researchers have previously debated as to which of these skills was the most important, today most take a more nuanced view, assuming that conceptual and procedural knowledge enhance each other. That is, as a child's conceptual knowledge grows, his procedural skill improves, and vice versa. The exact relationship between the two may vary across mathematical topics, but conceptual and procedural knowledge are both important and function together to contribute to a child's mathematical knowledge (Baroody, 2003; Rittle-Johnson & Siegler, 1998).

METACOGNITIVE PROCESSES

Metacognition can be defined loosely as "thinking about one's own thinking." Most theories distinguish between two types:

- **metacognitive knowledge** – what we know about our own thinking; also, how, when, and why to use particular strategies and resources; and
- **metacognitive regulation** – how we use what we know to regulate and control our thinking (Schraw & Moshman, 1995).

Thus, children engage in many metacognitive processes when they analyze problems, select appropriate strategies to solve them, regulate their problem-solving process, and check the validity of their answers.

Metacognitive processes are touched on throughout this research summary. We will focus particularly on self-regulation: the ability to set goals, plan, self-monitor, evaluate, learn adjustments, and choose a strategy (NMAP, 2008). Efforts to improve a child's self-regulation skills include prompting children to check their answers, set goals for improvement, and chart their daily progress. These efforts have been shown to also improve mathematics learning (e.g., Fuchs et al., 2003).

Self-Regulation

Self-regulation involves both motivation and cognitive processes, and is related to children's use of strategies for problem solving. As Siegler (1996) stated in the "overlapping waves" theory of development, children know and tend to use a variety of strategies for solving problems. Individual children choose different strategies for particular problems or in particular situations depending on differences in their knowledge of answers to problems and also their degree of perfectionism. Children who are not able to self-regulate effectively often have a poor knowledge of how to use strategies and may guess at the answer to a problem. As a result, they may be more likely to be labelled as "mathematics disabled" or to fail a grade (Siegler, 1988; Kerkman & Siegler, 1993). Expert problem solvers with good self-regulation skills, on the other hand, "spend more time analyzing problems before initiating solutions, reflect more frequently on their problem solving, and alter their approach more flexibly" (Fuchs et al., 2003, p. 307).

Consider the following example:
Marie-Eve and Emily are playing a game with connecting squares and a die. The aim of the game is to create the longest chain by taking turns rolling the die and adding the rolled number of squares to the chain. On her second turn, Marie-Eve has a chain of five squares and rolls a four. She pauses, looks at her chain, then adds a chain of four. Realizing that she already has five links, she counts them as "5" and counts on "6, 7, 8, 9" to find out how many she has in total. Emily rolls a six and quickly adds six more squares to her chain of three. She then proceeds to count all the squares: "1, 2, 3, 4, 5, 6, 7, 8, 9."

In this simple illustration, Marie-Eve has a more efficient strategy: counting on from a number she already knows instead of counting all of her tokens over again. In contrast, Emily has used a less efficient strategy, even though she arrives at the correct solution. Siegler (2000) suggests that earlier strategies like Emily's persist in children's repertoire because children may not be able to carry out more efficient methods. In addition, children who do not take the time to reflect on a problem may fail to see the inefficiency of a particular solution.

SOCIAL AND MOTIVATIONAL INFLUENCES

Children's Learning Goals

One of the most widely adopted theories of motivation for learning describes people based on their reasons for pursuing challenges and facing obstacles. Their goals may be either mastery-oriented or performance-oriented (Ames, 1992). Children with mastery-oriented goals tend to choose more difficult materials in order to challenge themselves. Their focus is inward, on their own learning, rather than outward, on a comparison to other people's performance. They attribute any problems to their own lack of effort and try harder the next time they face a challenge. In contrast, children who have performance-oriented goals focus primarily on comparing their abilities to others' and tend not to seek out challenges for themselves. If these children experience difficulty with a problem, they are more likely to give up. They tend to blame failure on their own lack of ability and avoid difficult problems in the future (Ames, 1992; Ames & Archer, 1988). Children with mastery-oriented goals perform significantly better in math than do students with performance-oriented goals (e.g., Gutman, 2006; Linnenbrink, 2005; Wolters, 2004).

So how can we, as educators, support children's learning if their own motivations affect their learning so strongly? Fortunately, children are not born with an unchangeable orientation to either mastery or performance goals. Like working memory, a child's goals can be acquired and encouraged through certain kinds of learning situations. For example, the following conditions tend to foster mastery-oriented goals:

- providing meaningful reasons for engaging in a task and understanding it;
- promoting high interest and intermediate challenge;
- emphasizing gradual skill improvement; and
- arranging for novelty, variety, and diversity (Ames, 1992).

Motivation

Related to mastery- and performance-oriented goals is the concept of intrinsic and extrinsic motivation. A child with intrinsic motivation to learn has "the desire to learn for no reason other than the sheer enjoyment, challenge, pleasure, or interest of the activity," while children who have extrinsic motivation for learning make efforts in the hope of some external reward (NMAP, 2008, pp. 4-12; also Berlyne, 1960; Hunt, 1965; Lepper, Corpus, & Iyengar, 2005; Walker, 1980). Several studies have shown that intrinsic motivation is associated with academic achievement and learning (e.g., Lepper et al., 2005; Gottfried, Fleming, & Gottfried, 2001). Most children, in fact most people, have a mixture of intrinsic and extrinsic motivational factors that drive their learning goals. Awareness of the importance of both types of motivation can be helpful in guiding educators' choices of activities and rewards.

Children's Beliefs About Learning

A child's academic goals and motivation for learning both play an important role in their mathematics education, but there is also a great deal of research on how children's beliefs influence their academic success, in particular their beliefs about mathematics and the source of their own success in mathematics (e.g., Leder, Pehkonen, & Torner, 2002; Muis, 2004). If children develop positive beliefs about mathematics and math education, they will develop a productive "mathematical disposition," that is, they will see math as making sense, as useful and worthwhile. They will feel that putting effort into their studies of math will pay off (National Research Council [NRC], 2001).

Without a positive and productive mathematical disposition, children are likely to believe that they "can't do math," that they are not naturally mathematically minded, and thus that they will never succeed in math, regardless of the amount of effort they put in. Children who believe that effort is necessary to do well in math will persist longer on complex tasks than children who believe that success depends on having innate ability (NMAP, 2008). In general, children who believe that intelligence is malleable and who put in effort academically tend to do better in school than those children who believe intelligence cannot be changed (Dweck, 1999). It is important for educators to pay attention to children's beliefs about the nature of intelligence since, fortunately, these beliefs can be changed. Greater emphasis on the importance of effort leads to greater engagement in math and thus to improved achievement (Blackwell, Trzesniewski, & Dweck, 2007; NMAP, 2008).

Self-Efficacy

The term "self-efficacy" refers to the set of beliefs one has about one's own ability to succeed at difficult tasks (Bandura, 1997). Self-efficacy correlates significantly with performance in mathematics for students from elementary school to university (e.g., Pajares & Miller, 1994; Kloosterman & Cougan, 1994). In their early years and the primary grades, children's beliefs about mathematics are not related to their achievement, since most children at this age see themselves as able to do math. This confidence decreases over the years, however. By Grade 6, student beliefs are correlated with their achievement: children do as well or as poorly as they believe they are capable of doing, and low achievers start to dislike mathematics (Kloosterman & Cougan, 1994; Wigfield et al., 1997). Ability is important for success in mathematics, but feelings of self-efficacy influence how ability is expressed in actual performance.

MATH ANXIETY

Some people experience "math anxiety," an emotional reaction in situations that involve numbers, ranging from a mild apprehension to a genuine fear or dread (NMAP, 2008). Not only is math anxiety stressful, it is also related to low performance in mathematics, avoidance of more advanced studies in math, and poor scores on standardized tests. Little is known, however, about how math anxiety begins or what the contributing factors are (NMAP, 2008). Although conventional wisdom says that girls are more anxious about mathematics than boys, research has found that gender has little effect on math anxiety overall (e.g., Ashcraft & Ridley, 2005). In some studies, girls in all grades have reported higher levels of anxiety about mathematics, but their anxiety does not seem to translate to either mathematics performance or the degree to which they avoid math (Hembree, 1990). It has been suggested that girls may simply be more willing to admit anxiety (Ashcraft & Ridley, 2005).

Recent research on math anxiety has shifted from an investigation of contributing factors to a more process-oriented approach. Studies have tried to understand the cognitive consequences of math anxiety. It was discovered that people with math anxiety may have a difficulty with working memory. The hypothesis is that their working memory capacity is occupied with managing their anxiety, instead of trying to solve the mathematics problems (Ashcraft & Kirk, 2001; LeFevre, DeStefano, Coleman, & Shanahan, 2005). When children's anxiety is reduced, often through some kind of cognitive therapy, their math achievement improves, often more than either the children or their teachers expected. It appears that their ability had been depressed by their own anxiety (NMAP, 2008).

Changes can be made in the classroom to reduce math anxiety. Some changes, such as providing calculators, have not proved effective. On the other hand, a review of basic skills and a focus on the relationship between good study habits and performance have shown positive effects for students with math anxiety (Hutton & Levitt, 1987). Further, if students are encouraged to attribute success to controllable factors, like hard work and test preparation, they tend to work more persistently and their performance improves (Dweck, 1975).

Some students may be more likely than others to become anxious about mathematics. Risk factors include low mathematics aptitude, low working memory capacity, concern over public embarrassment, gender, and negative attitudes in significant adults (educators and parents). In addition, social and intellectual support from peers and teachers is associated with better performance in mathematics for all students, regardless of whether they have math anxiety (NMAP, 2008).

TRANSFER OF LEARNING

To achieve success in mathematics, it is essential to be able to **transfer** skills from one type of problem to another. This means being able "to correctly apply one's learning beyond the exact examples studied to superficially similar problems (near transfer) or to superficially dissimilar problems (far transfer)" (NMAP, 2008, p. 7). Children are more likely to achieve transfer if they have a deeper conceptual understanding of the material, which is often achieved through work with more difficult problems. Challenging material requires children to apply more attention and effort to process the information, which leads to better retention (NMAP, 2008). Abstract representations of information can also benefit transfer of learning to more concrete examples (e.g., Sloutsky, Kaminski, & Heckler, 2005; Uttal, 2003). However, children need to start by working with less challenging material in order to get an initial understanding. Only then will work with more challenging material allow them to deepen their understanding and improve their ability to transfer learning.

INTERWOVEN SKILLS FOR MATHEMATICAL COMPETENCE

As we have seen, there are a number of cognitive and social/emotional factors that contribute to achieving success in mathematics. All these factors interact and affect one another as the process of mathematics learning develops over time. Some models have been put forward by researchers to frame how this process happens. For instance, the Competence, Learning, Intervention, and Assessment (CLIA) model states that mathematical competence can be reached only if children gain five skills:

- **mathematical knowledge** (e.g., facts, symbols, algorithms[a], concepts, and rules)
- **heuristic methods** (e.g., systematic strategies for problem solving)
- **metaknowledge** (e.g., thinking about one's own thinking, emotions, and motivation)
- **self-regulatory skills** (e.g., planning, monitoring)
- **self-efficacy beliefs** (e.g., thinking about oneself in relation to mathematics)

These skills all develop concurrently, not one after the other (De Corte & Verschaffel, 2006).

It has been suggested that children need to have math made relevant to them; in particular, they need to have opportunities to use the knowledge and skills they have learned to solve problems. However, exposure and practice are not enough: they must also want to use the knowledge and skills they have learned. All of these conditions are influenced by the child's beliefs, not only about what he finds interesting, but also about what counts as a mathematical context (Perkins, 1992). Thus, in addition to skills and strategies, one's beliefs and attitudes are also important.

THE IMPORTANCE OF GOOD MATHEMATICS TEACHING

An important predictor of children's achievement is the quality of the early years educator and of the classroom teacher (e.g., Darling-Hammond, 2000; Darling-Hammond & Youngs, 2002; Hanhushek, Kain, & Rivkin, 1998). Studies on school-aged children have shown that effective teaching can account for the greatest differences between more and less effective schools (Clotfelter, Ladd, & Vigdor, 2007; Klecker, 2007). The National Mathematics Advisory Panel[b] stated that "teachers are crucial for creating opportunities for students to learn mathematics" and that in a single elementary school year, "differences in the quality of teaching [can] account for between 12 and 14% of the total

[a] In mathematics, an algorithm is a set of precise step-by-step instructions for how to arrive at an answer. It refers to a formal procedure, usually one that is explicitly taught.

[b] In 2006, the National Mathematics Advisory Panel (NMAP), a panel of 24 distinguished mathematics researchers, was convened to advise the President of the United States and the U.S. Secretary of Education on ways to foster increased mathematics performance using research-based instructional methods (NMAP, 2008). The panel produced their report in 2008. One of the strongest of the panel's recommendations concerned the importance of applying what is known from research on how children learn to the teaching of mathematics. In particular, the panel noted that "a) there are great advantages for children who have a strong start in mathematics, b) conceptual understanding, procedural fluency, and quick, effortless (i.e., "automatic") recall of facts are related in a mutually reinforcing and beneficial way, and c) effort, not just inherent talent, is a vital component of achievement in mathematics" (NMAP, 2008, p. 11).

variability in students' mathematics achievement gains" (NMAP, 2008, p. 35). This effect is compounded when students have several either effective or ineffective teachers one after the other.

Truly effective mathematics teaching brings together four required components:

- an appreciation of the discipline of mathematics itself, and of what it means to do mathematics
- an understanding of how children learn
- the provision of a problem-solving environment for learning
- the integration of assessment into teaching to enhance both learning and instruction (National Council of Teachers of Mathematics [NCTM], 1989).

A variety of educator characteristics can positively affect children's performance in mathematics. First and foremost, the educator's own knowledge of the subject significantly influences children's learning; this relationship appears to be particularly true for mathematics (Wayne & Youngs, 2003). In addition, math teachers who had continued to study mathematics after high school had students with greater mathematical gains than those students whose teachers did not hold advanced degrees, whether these degrees were in mathematics or not (Hill, Rowan, & Ball, 2005).

Although researchers agree that educators' knowledge of their subject matter contributes to children's learning, it takes more than knowledge to be effective. The way educators put their knowledge into action plays a vital role in the development of children's understanding of mathematics. For example, an educator's mathematical behaviours – such as their level of explanation, their choice of representation, and their interactions with students' mathematical thinking – all influence children's mathematics performance (Hill et al., 2005). Experience as an educator is also a factor, but its impact is influenced by the qualities and abilities of the individual teacher (Kukla-Acevedo, 2009). The importance of strong mathematics educators, in preschool and at all grade levels, cannot be overstated.

SUMMARY

This section has provided information on some of the mental processes underlying children's achievement: attention, working memory, and the retrieval, transfer, and retention of information; factual, conceptual and procedural knowledge; strategy selection and use; and self-monitoring behaviours. Knowledge of these processes can help educators understand how children learn and thus how they can best support that learning. While these cognitive building blocks are important, other factors also contribute to children's mathematics achievement. Motivation, self-regulation, and mathematics anxiety all can have a strong effect on children's cognitive processing, and thus on their achievement. Effective educators must take all of these factors into account.

PART 2: THE DEVELOPMENT OF MATHEMATICS CONCEPTS

EARLY MATHEMATICAL ABILITIES

Young children have a natural desire to understand the world around them. They are "active, resourceful individuals who can construct, modify, and integrate ideas by interacting with the physical world and with peers and adults" (NCTM, 2000, p.75). Mathematics is one means by which we understand the world, and children engage with math long before they begin school (Bryant, 1997).

Clements (2004) asserts that even before Kindergarten, "children have the interest and ability to engage in significant mathematical thinking and learning" (p. 11). During the early years, children explore the mathematical dimensions of their world, comparing quantities, finding patterns, navigating their environment, and tackling real problems (National Association for the Education of Young Children [NAEYC] & NCTM, 2002). Knowledge of quantity emerges early in life and develops significantly during a child's first three years. Research has shown that infants can tell the difference between small quantities, for instance, between two items as opposed to three items (Starkey, Spelke, & Gelman, 1990). Toddlers typically learn their first number word (usually "two") at around twenty-four months. By age four, children are able to compare quantities and use words like "more" and "less." As children get more experience with counting, they begin to count larger collections, count on from a given number, and learn number patterns. Children also explore shape, space, and measurement. A child building a tower out of blocks is using knowledge about shape (which blocks are best for the base of the tower), space (where best to place the blocks to ensure a sturdy tower), and measurement (how many blocks can be placed on the tower before it is taller than the builder). Pre-Kindergarten children are also keen to recognize and analyze patterns – the beginnings of algebraic thinking (Clements, 2004).

"Research suggests that children's early mathematical experiences play an enormous role in the development of their understanding of mathematics, serve as a foundation for their cognitive development, and can predict mathematics success in the high school years" (Shaklee, O'Hara, & Demarest, 2008, p. 1). The National Council of Teachers of Mathematics (NCTM) also maintains that the foundation for children's mathematical development is established in the early years (2000). Moreover, NCTM, in a joint positional statement with NAEYC, asserted that children aged three to six require high quality, challenging and accessible math education in order to build a strong foundation for their future mathematics learning (2002).

Children's mathematics ability at the beginning of Kindergarten is a strong predictor of later academic success, even stronger than their early reading ability (Duncan et al., 2007). Mathematics ability is, in turn, based on knowledge accumulated during the years before Kindergarten. Children learn by building on prior knowledge, extending as far back as early childhood. A theory of "overlapping waves" of learning and development describes the gradual, incremental processes that occur as children grow and learn (Siegler, 1996). Studies observing children at play reveal that young children naturally engage in a significant amount

of mathematical activity (Clements & Sarama, 2005; Ginsburg, Inoue, & Seo, 1999; Seo & Ginsburg, 2004). Even before they begin elementary school, children can reason and solve problems (Gopnik, Meltzoff, & Kuhl, 1999; NMAP, 2008).

In deciding what is "developmentally appropriate," we need to look beyond a child's age or grade. Both NRC and NMAP document the finding that what children are ready to learn is largely a result of their prior opportunities to learn (Duschl, Schweingruber, & Shouse, 2007). Claims that children are either too young, in the wrong stage, or not ready to learn something have been shown time and again to be wrong (NMAP, 2008). A significant body of research has shown that young children are more competent than was previously thought. Moreover, this research suggests that without adequate attention to math in the early years, a child may be at risk for later school failure (Lee & Ginsburg, 2007).

There is strong evidence for the importance of a well-built foundation in mathematics, just as there is for reading. Sarama and Clements (2004) argue that a complete mathematics program may also contribute to children's later learning of other subjects, especially literacy. Much of the recent research has reported that mathematics does in fact support the development of literacy.

Numerosity and Ordinality

Children are born with some abilities necessary for processing quantities, abilities that have also been noted in rats, pigeons, and other primates. They can make decisions about which quantity is more or less, and can, to some degree, understand processes such as "taking away, resulting in less." Researchers disagree about the connection between these abilities and actual mathematical understanding; nonetheless, it is clear that infants and very young children are capable of more than was once assumed. For instance, studies of six-month-old infants have shown that they can tell the difference between larger and smaller quantities. They can do this both with objects they see and with sounds they hear. However, this ability is limited and they are more accurate with smaller quantities. When they are asked to compare two sets that both contain a large number of items, they can only recognize the difference when the larger set contains at least double the number of items as the smaller set (Brannon, Abbott, & Lutz, 2004; Lipton & Spelke, 2003; Xu & Spelke, 2000). When dealing with smaller numbers, babies aged four- to seven-and-a-half months can discriminate between a set of two and a set of three objects, but not between a set of four and a set of six objects (Starkey & Cooper, 1980).

The technical term for "the ability to discriminate arrays of objects based on the quantity of presented items" is numerosity (Geary, 2006, p. 780). Sensitivity to numerosity has been demonstrated many times using one to three objects, and sometimes four, with infants, even as early as the first week of life (e.g., Antell & Keating, 1983; Starkey, 1992; Starkey, Spelke, & Gelman, 1983, 1990; van Loosbroek & Smitsman, 1990). These findings suggest that even as infants, we have an intuitive sense of approximate magnitude (i.e., how much there is) called ordinality (Dehaene, 1997; Gallistel & Gelman, 1992). This sense of more and less emerges in a very basic form around ten months of age (Brannon, 2002; Feigenson, Carey, & Hauser, 2002).

Arithmetic

As we discussed above, two important cognitive factors that affect learning are the mental representation of information and the memory for information. Research on these factors has been done with children aged one-and-a-half to four. The findings show that at age two, children can mentally represent and remember one, two, and sometimes three items. By two and a half, their representation of and memory for up to three items is more consistent (Starkey, 1992). By age three or three-and-a-half, up to four items can be represented and remembered. In the same study, the children's addition and subtraction abilities were also examined, using a nonverbal calculation task. The youngest children, who were one-and-a-half, understood addition and subtraction with numbers less than or equal to two (e.g., 1 + 1; 2 - 1), but not with larger numbers. Two-year-olds were accurate with values up to three, and none of the children (even up to age four) were accurate with values of four or five (Starkey, 1992). Research such as this suggests that between ages two and three, children are not just aware of the concept of small numbers, but can also begin to learn how to solve simple nonverbal calculations involving one and two items. By the time they are four, many children can solve problems involving three (and sometimes four) items (Jordan, Huttenlocher, & Levine, 1994).

These basic arithmetic problems can be made slightly more complex for older preschool children by introducing the concept of the inverse relation between addition and subtraction. For example, when starting with two items, if one item is added and one item is taken away, there are still two items left; adding one and taking away one cancel each other out. In the example "2 + 1 – 1 = ?", if this inverse relation is understood, no adding or subtracting needs to be performed to know that the answer is 2. In studies, some four-year-olds used a procedure based on the inverse relation between addition and subtraction to solve problems like this. At this young age, they demonstrated at least a basic understanding of this fundamental principle of arithmetic (Klein & Bisanz, 2000).

Number Concepts

Sometime between ages two and three, children begin to map the number words of their language and culture onto their knowledge of numerosity and systems of magnitude, beginning with counting (Spelke, 2000; Gelman & Gallistel, 1978). Children appear to know very early that the number words all represent different quantities and that the sequence in which they say these number words is important (Gelman & Gallistel, 1978). At the same time, they also understand that number words are different from other descriptive words, such as "big" or "red" (Geary, 2006). Children may know certain qualities of numbers before they are able to apply and use that knowledge fully. For instance, by age two and a half, children can tell the difference between a set of three items and a set of four items. They also know that "4" is more than "3." However, they still may not be able to consistently connect number words with quantities in order to label sets as containing three or four items (Bullock & Gelman, 1977). It has been argued that at least a year of counting experience, usually from age two to age three, is necessary for children to both associate number

words to their mental representations of quantities and use that knowledge in counting (Wynn, 1992). Quantities of four and above seem to be more difficult for preschool children.

Counting Procedures

Five implicit principles are thought to guide a preschool child's development of counting procedures (Gelman & Gallistel, 1978):

- **Stable order** refers to the fact that the number words are always used in the same order (e.g., counting in the order of "1, 2, 4" is incorrect).
- **One-to-one correspondence** means that one and only one number word can be assigned to each counted object in the set (e.g., an item in a set that has been assigned "3" cannot also be assigned "5").
- **Cardinality** refers to the fact that the value of the last number word used when counting indicates the quantity of items in the set (e.g., counting "1, 2, 3, 4" means there are four items in the set).
- **Abstraction** means that any set of items can be counted (e.g., a book, two bananas, and three pencils can be counted together as a set of six items).
- **Order irrelevance** means that items can be counted in any order (e.g., counting from right to left, left to right, or in no particular sequence at all will result in the same total number of items).

The first three principles are the basic "how to count" rules, which set the initial structure for children's developing knowledge of counting (Gelman & Meck, 1983). Children refine their understanding of counting and add to these basic principles as they observe and think about counting. For awhile, children assume that some aspects of counting are essential when in fact they are just conventions. For instance, by habit, we may always count from left to right. Observing this, children may believe that counting must necessarily be done in a standard direction. We also, by habit, tend to move from one item to the item next to it when we count. Children may gather from this that counting must be done in this way to be accurate, and that adjacency is an essential element of counting. By age five, most children know the essential features of counting but many continue to believe that adjacency is mandatory (LeFevre et al., 2006).

By age five, most children's knowledge of the essential principles is quite good, though they still make some mistakes. By the end of Kindergarten, many children can count sets that contain a quantity of items for which they know the number words: if they know the numbers up to twelve, they can accurately count a set with twelve items. However, quite a few children are still struggling even in higher grades. Those who are not proficient counters and who do not know the essential principles by the time they enter Grade 1 may be at risk for difficulties with mathematics (Geary, 2003).

Geometry and Measurement

Geometry and measurement have been called the second most important area of mathematical learning. According to some authors, "one could [even] argue that this area – including spatial thinking – is as important as number" (Sarama & Clements, 2009, p. 159). Geometry and measurement are important partly because they make real-world connections: "Geometry, measurement, and spatial reasoning are important… because they involve 'grasping'… that space in which the child lives, breathes, and moves… that space that the child must learn to know, explore, and conquer in order to live, breathe, and move better in it" (NCTM, 1989, p. 48).

Geometry and measurement also contribute to the foundation for learning in math and other subjects (Clements, 2004). For example, spatial thinking is essential to the development of number quantification, non-routine problem-solving ability, and mathematical reasoning (Sarama & Clements, 2009). In addition, spatial thinking provides a real-life application for number and arithmetic (Clements & Stephan, 2004).

Children in preschool and Kindergarten should be given rich opportunities to explore shape (Clements, 2004; Clements & Sarama, 2000; Clements & Stephan, 2004). They can also benefit from activities that encourage them to consciously reflect on the properties and attributes of shapes (Orton, Orton, & Frobisher, 2005). Such activities may include sorting, finding examples of shapes in the environment, combining shapes to create new ones, and constructing and altering shapes.

When they enter school, children commonly have a working knowledge of shape, congruence, and symmetry; instruction should be designed to "build on this knowledge and move beyond it" (Clements, 2004, p. 285). Children's formal geometric knowledge and skills will benefit from being exposed to basic geometric shapes, names, and other concepts; however, by this point, mere exposure is insufficient. Children "must eventually transition from concrete (hands-on) or visual representations to internalized abstract representations" (NMAP, 2008, p. 29).

THE TRANSITION TO SCHOOL

Around the age of four or five, children may be receiving more formal education in mathematics in preschool or Kindergarten. They bring with them their intuitive understanding of quantities and accumulated experiences with mathematical concepts in their daily life. At first, the more formal mathematics lessons tend to be disconnected from children's intuitive understandings, but gradually, they will achieve integration of these two systems, their formal and informal learning.

Number Concepts and Counting

By the time they reach Kindergarten at age four or five, most children can use number words to solve simple addition and subtraction problems with small numbers (Baroody & Ginsburg, 1986; Groen & Resnick, 1977; Saxe, 1985; Siegler & Jenkins, 1989). At this stage, they often solve problems by using concrete objects (including fingers) to help them count (Geary, 2006). These tools serve to connect numerosities in the world to internal representations, they reduce the load on memory, and they help make sure the procedures are carried out correctly (Siegler & Shrager, 1984).

Cardinality and Ordinality

As mentioned above, cardinality refers to the fact that the last number counted is the total number of items in the set. Ordinality, at its most basic level, is the concept of more and less. A child's sense of ordinality develops into an understanding that higher numbers are associated with more items and lower numbers with fewer items. Children need to understand both cardinality and ordinality to become competent in math. Without cardinality, counting would not provide any meaningful information, and "without [ordinality], distinct numerosities such as 'one' and 'four' bear no more relation to one another than do cows and blenders" (Brannon & Van De Walle, 2001, p. 54).

When children know the sequence of number words (1, 2, 3, 4...), they can then develop more precise mental representations of numbers beyond three or four, which in turn enhance their knowledge of cardinality and ordinality. In fact, children seem to have an implicit understanding of both cardinality and ordinality even before they learn the sequence of number words (Bermejo, 1996; Brainerd, 1979; Brannon & Van de Walle, 2001; Cooper, 1984; Huntley-Fener & Cannon, 2000; Ta'ir, Brezner, & Ariel, 1997; Wynn, 1990, 1992). However, it is not enough to have an intuitive sense of these concepts. To become mathematically competent, it is essential to have a more mature sense of cardinality and ordinality in which they are connected to the counting sequence, and this takes time.

Here is an example of how a preschooler who has an immature understanding of cardinality might act. Imagine that an educator counts aloud the fingers on a child's hand and asks him how many fingers he has. He may need to count his fingers again before answering. Only a child with a mature grasp of cardinality will be able to simply repeat the last number word that the adult said. By five years old, most children have a good grasp of cardinal value for quantities of ten and under, so they are able to do this (Bermejo, 1996; Freeman, Antonucci, & Lewis, 2000). Once a child has learned the verbal counting system, their abilities improve rather dramatically toward a mature appreciation of cardinality (Brannon & Van De Walle, 2001).

Counting

As we have seen with respect to children's understandings of ordinality and cardinality, counting is a foundational component of children's early work with number (NCTM, 2000). It takes some time to master the verbal counting system. Typically, it begins to emerge in children around the age of four and solidifies by age five or six, by which time children generally make few errors and have a good grasp of the essential counting principles (Schaeffer, Eggleston, & Scott, 1974; Wynn, 1990, 1992). By the age of five, most children have acquired the basic foundational skills of numeracy: they can match sets that contain the same number of items, label small numerosities, and use counting to determine cardinality (e.g., Bermejo, 1996; Bermejo & Lago, 1990; Fuson, 1988; Gelman & Gallistel, 1978; Huttenlocher et al., 1994; Mix, 1999; Mix, Sandhofer, & Baroody, 2005; Wynn, 1990). When children start school, their knowledge of number is informal, but nonetheless rich and varied (Baroody, 1992; Fuson, 1998; Gelman, 1994). During the early elementary years, teachers help students to strengthen their sense of number by moving them from basic counting techniques to a more sophisticated understanding of numbers (NCTM, 2000).

Commutative and Associative Properties

A solid sense of numbers includes understanding that numbers are sets of smaller numbers that can be decomposed and recombined. For example, 12 can be decomposed into $2 + (4 + 6)$ and recombined into $(2 + 4) + 6$, or decomposed into $2 \times (3 \times 2)$ and recombined into $(2 \times 2) \times 3$.

Decomposition and recombination are related to two properties of the operations of addition and multiplication.

- the **commutative property** refers to the fact that you can change the order in which you add or multiply two numbers without changing the answer. For example, a child who understands the commutative property of arithmetic knows that the sum of $4 + 3$ is the same as the sum of $3 + 4$. In the same way, 5×8 and 8×5 have the same product. More generally, the commutative property can be stated as "$a + b = b + a$" and "$a \times b = b \times a$."
- the **associative property** is similar to the commutative property, but deals with more numbers. It states that the order in which three numbers are added or multiplied does not affect the sum or product. This can be stated as "$a + (b + c) = (a + b) + c$" and "$a \times (b \times c) = (a \times b) \times c$."

Some work has been done on the associative property of addition (Canobi, Reeve, & Pattison, 1998, 2002); however, most of the research in this area has focussed on the commutative property of addition (Baroody, Ginsburg, & Waxman; 1983; Resnick, 1992). Resnick (1992) proposed that knowledge of the commutative property is built on conceptual steps. First, in preschool or Kindergarten, children go through a pre-numerical stage during which they solve problems by manipulating physical objects. They find out that it doesn't matter in what order objects are combined into a set, because the total will still be the same (Gelman & Gallistel, 1978; Resnick, 1992). Next, around four or five years old, children begin to map specific quantities onto this action, for example, five cars plus three trucks equals three trucks plus five cars (Canobi et al., 2002; Sophian, Harley, & Martin, 1995). Following this, when they are in Grade 2 or 3, children move away from a reliance on physical objects and begin to use only numbers: $5 + 3 = 3 + 5$ (Baroody et al., 1983). Finally, children achieve a formal knowledge of the commutative property as an arithmetic principle ($a + b = b + a$); the exact timing of this last stage is not certain (Resnick, 1992).

Understanding of the associative property of addition is acquired in much the same way, beginning in Kindergarten with physical objects and moving to an implicit understanding in Grade 1 or 2. However, children do not come to understand the associative property as an arithmetic principle until they have an implicit understanding of the commutative principle (Canobi et al., 1998, 2002).

The Mental Number Line

Mathematics involves cognitive processes that require the dual coding of imagery and language. Imagery is fundamental to the process of thinking with numbers because it allows us to create mental representations for mathematical concepts (Bell & Tuley, 2003). One of the most important of these representations is the **mental number line**. Some of the central achievements of

formal mathematics depend on understanding the relationship between number and space; fundamental to this is the arrangement of numbers on a line (de Hevia & Spelke, 2008).

Learning the concept of number itself appears to be related to a child's ability to generate a mental number line (Dehaene, 1997). A mental number line is an imaginary horizontal line with numbers along it in ascending order. It is, of course, a metaphor, not an actual structure in the brain. This number line is a mental image that reflects our knowledge, a tool we use to represent numbers and relative magnitudes.

Forming a mental number line requires the ability to visualize and abstract number to order numbers by quantity, to locate a given number along a line, and to generate any portion of the number line that may be required for problem solving (Gervasoni, 2005). It is related to learning addition and subtraction, as well as the estimation of the magnitude of numbers (Siegler & Booth, 2005).

There are three main areas in which the number line is particularly useful for young children's mathematical development (Griffin, Case, & Siegler, 1994). First, a mental number line allows children to respond to questions about relative magnitude without referring to concrete objects. Second, mental number lines support the acquisition of the increment rule, which describes how addition or subtraction alters the cardinal value of the set and therefore moves that value up or down on the number line. Third, children who have developed a mental number line can also determine the relative position of a number on that line, which is useful for determining relative quantity when it cannot be determined more directly (Gervasoni, 2005).

The ability to use the mental number line to represent specific quantities only emerges with formal education, after the transition to school (e.g., Siegler & Opfer, 2003). Research indicates that young children have difficulty making estimates of the position of a number on the number line (e.g., placing 84 on a number line from 1 to 100), but that this skill improves over the elementary years (Siegler & Booth, 2004; Siegler & Opfer, 2003). Their initial difficulty may occur because young children tend to see the distance between 1 and 2 as larger and more certain than the distance between 51 and 52; the numbers get "squished up" towards the right end of the number line (Dehaene, 1997; Gallistel & Gelman, 1992). By Grade 6, most children have a correct, linear sense of the number line and of the fact that numbers are spaced evenly along it (Siegler & Opfer, 2003).

THE ELEMENTARY YEARS

After the transition to the more formal education system of elementary school, children solidify the knowledge gained earlier and deepen their conceptual understanding of mathematics. They face new challenges, such as fractions, and they use new skills, such as estimation, problem-solving strategies and algorithms. They also achieve understanding of new concepts, such as arithmetic operations, proportion, reversibility, and commutative and associative properties. Because of the breadth and extent of foundational skills that need to be mastered, academic success in mathematics can be challenging.

Biologically Primary and Secondary Knowledge

As we have discussed, for very young children mathematics-related thinking is primarily made up of inherent types of cognition, such as language and early quantitative competencies. These have been called biologically primary abilities because they typically emerge with little or no formal instruction. They appear universally, across all cultures. Once children have reached school age, however, they build on these biologically primary abilities to learn skills that need to be formally taught. Some of the information learned in school is considered to be a "cultural invention," with arbitrary symbols such as number words. This knowledge is referred to as biologically secondary (Geary, 1994, 1995).

An example of socially constructed, biologically secondary information is the base-10 system, an essential component of mathematics. A child who does not grasp the fundamentals of this system will have difficulty understanding other concepts (Geary, 1995). As Geary (2006) states: "Many children require instructional techniques that explicitly focus on the specifics of the repeating decade structure of the base-10 system and [techniques] that clarify often confusing features of the associated notational system" (p. 791). Children who speak certain European languages may need more help with this than children who speak Asian languages. In Chinese, for instance, the base-10 system is made obvious by the number words for 11, 12, and 13, which translate as "ten-one, ten-two, ten-three." This contrasts with the English "eleven, twelve, thirteen," which make no reference to base-10 (see, for example, Fuson & Kwon, 1991). This transparent connection between the number word, the Arabic digit, and the magnitude represented gives children who speak Asian languages an initial advantage over English-speaking children in understanding the base-10 concept (Miller et al., 2005). It may also enhance their conceptual understanding of arithmetic (Miura, 1987), although children who speak non-Asian languages appear to catch up quickly. Teachers can benefit from knowing about possible difficulties in linking numbers to corresponding words. Children in French immersion classes, for example, may be confused about the numbers between 70 and 100 (e.g., compare soixante-quinze [translates as sixty-fifteen] to seventy-five) even when they have mastered the labels in English (Seron & Fayol, 1994).

Fractions

To build their knowledge and understanding of fractions, elementary school children need to already have a firm base of skills and concepts. They need to have learned and practised certain basic arithmetic facts until they come automatically. They must be able to perform mathematical procedures with whole numbers and possess a deep understanding of core mathematical concepts (NMAP, 2008). Procedural and conceptual skills also influence a child's ability to estimate, make computations, and to find the solution to word problems.

Children have considerable difficulty learning the conceptual and procedural aspects of fractions (Geary, 2006). Research has focussed on these aspects of fractions (Clements & Del Campo, 1990; Hecht, 1998; Hecht, Close, & Santisi, 2003) and on the mechanisms that influence their acquisition (Miura, Okamoto, Vlahovic-Stetic, Kim, & Han, 1999; Rittle-Johnson,

Siegler, & Alibali, 2001). At first, when children begin to learn the formal features of fractions, such as the numerator and denominator system, they tend to rely on what they already know about whole number counting and arithmetic (Gallistel & Gelman, 1992).

Although fractions are considered biologically secondary information in a formal mathematics context, children already have some understanding of part/whole relationships based on their experience with physical objects (Mix, Levine, & Huttenlocher, 1999). In preschool and early elementary school, children already understand simple fractional relationships – they know whether a cookie is being divided equally, or if one person is receiving a larger share. It is not known yet if the ability to visualize parts of a whole is a biologically primary ability (Geary, 2006).

Research has focussed on older elementary school children's computational skills, conceptual understanding, and ability to solve word problems that involve fractions (Byrnes & Wasik, 1991; Rittle-Johnson et al., 2001). Once a child has conceptual knowledge of fractions, that knowledge will likely have an effect on problem-solving performance. As with whole numbers, procedural knowledge will also inform conceptual knowledge when learning fractions (NMAP, 2008). Children's procedural ability has been shown to predict computational skills, and computational skills in turn predict accuracy at solving word problems with fractions and estimation skills (Hecht, 1998). In addition, the acquisition of conceptual knowledge of fractions and basic arithmetic skills was related to children's working memory capacity and to the amount of time spent on the task in class (Hecht, 2003).

Number Sense

Number sense can be broadly defined as the understanding of number and operations, the ability to use this understanding to learn and develop strategies for handling numbers and operations, and the ability to use numbers as a way of communicating and dealing with information (McIntosh, Reys, & Reys, 1992). (Definitions vary slightly in curriculum documents of different Canadian provinces.) More specifically, **number sense** encompasses three subcomponents:

- knowing about and using numbers (e.g., number order, multiple representations, relative and absolute magnitude)
- knowing about and using operations (e.g., mathematical properties, such as the commutative and associative properties, and relationships between operations)
- knowing about and using numbers and operations in computational settings (e.g., use of estimation, knowing that multiple strategies exist for the solution of any problem, efficient use of problem-solving methods, reviewing and checking one's answer) (McIntosh et al., 1992).

As many mathematical skills, number sense is not achieved all at once, but rather is a process that unfolds over years, developing with age and experience. For older elementary students, the third subcomponent is the most relevant. We will therefore examine number sense as it relates to estimation, problem solving, and word problems.

Estimation

Estimation may not be a formal subject in elementary school, but it is a skill that people use frequently, both in and out of school. To estimate is to approximate the value of something, often when it is difficult or unnecessary to determine an exact answer. We also use estimation to check whether our calculation of an answer is reasonable. Sowder (1992) identifies three forms of estimation: computational (e.g., estimating the answer to a word problem), measurement (e.g., estimating the area of the classroom), and numerosity (e.g., estimating the number of people at a soccer game). Siegler and Booth (2004) added a fourth form, number line estimation (e.g., placing numbers 0-100 on a number line).

Research on estimation has focussed on computational arithmetic (Case & Okamoto, 1996; Dowker, 1997, 2003; LeFevre, Greenham, & Waheed, 1993; Lemaire & Lecacheur, 2002) and on work with the number line (Siegler & Booth, 2004, 2005; Siegler & Opfer, 2003). These studies have shown that children, and some adults, find it hard to make reasonable estimates. The skill of estimation appears only with formal schooling and requires practice. For all types of estimation, both children and adults use a variety of strategies. Their skills improve in efficiency, sophistication, and adaptivity with age and experience (De Corte & Verschaffel, 2006).

Problem Solving

Research in the fields of both mathematics education and cognitive science has shown the benefits of a standards-based curriculum for mathematics instruction (e.g., NCTM, 2000). Although traditional direct instruction techniques are helpful, students also benefit from developing their own strategies for problem solving. They are required to "convey their personal understandings of [a] problem so that they can choose between the relative merits of different strategies that they invent" (Moseley & Brenner, 2009, p. 2).

In this section, we will examine what the research says about problem solving with both arithmetic and word problems.

Arithmetic Problems

There is not just one way to solve a problem, and no student will use only one strategy to solve all problems. An individual child will use a variety of arithmetic strategies, even within the same day or for the same type of problem (see Siegler, 1998, for a review). It is common for people, from young children up to adults, to have multiple and flexible strategies when learning arithmetic – addition, subtraction, multiplication, and division (LeFevre, Smith-Chant, Hiscock, Daley, & Morris, 2003).

Older elementary school children sometimes use the algorithms and strategies taught in school when they do multi-digit arithmetic. However, researchers have found that they may also use varied, informal strategies that differ from what they have been taught (e.g., Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Reys, Reys, Nohda, & Emori, 1995). Both children and adults use this kind of invented strategy.

A research project tracked children's arithmetic problem solving for three years, and identified five categories of invented strategies, of which three were most frequently seen:

- **combining units** strategies, wherein the 100s, 10s, and units are dealt with separately, for example 37 + 38 = 30 + 30, then 7 + 8
- **sequential** strategies, wherein the value of the second number is counted up or down from the first number, for example 37 + 38 is solved by 37 + 30 = 67, 67 + 8 = 75
- **compensating** strategies, wherein the numbers are adjusted to simplify the arithmetic, for example, 37 + 38 = (35 + 35) + 2 + 3 = 75) (Carpenter et al., 1998).

Researchers noted that students who tended to use their own methods would typically sample from all three of these strategy categories. In general, these students could transfer their knowledge to new and different problems better than those who followed only the standard step-by-step procedures they had been taught.

Children's choice of strategy often depends on the amount of conceptual knowledge they have – for instance, their knowledge of addition, units, grouping by tens, and the properties of the four basic operations (Ambrose, Baek, & Carpenter, 2003). This is another instance of the way conceptual and procedural knowledge influence each other.

Regardless of the type of strategy used, whether invented or taught, the flexible and adaptive use of multiple strategies is a characteristic of expertise with multi-digit arithmetic (De Corte & Verschaffel, 2006). As discussed in the section on students' beliefs about learning, the way students feel about math is an important factor in their success or failure in the subject. Research shows that the way students approach problem solving is also a factor. For instance, it is unproductive to "stubbornly" use standard step-by-step procedures in cases where mental arithmetic would be more appropriate, for example, to calculate the problem 4,002 – 3,998 (e.g., Buys, 2001). Students' fear of taking risks in problem solving will also influence their ability to succeed (Thompson, 1999).

Word Problems

Students in the early elementary grades encounter three general types of one-step word problems:

- **change problems** contain some event that changes the value of a quantity: Robin has 5 pencils and Carly gives him 3 more; how many does Robin have now? Change problems can be subdivided into two categories, depending on whether the quantity increases or decreases.
- **combine problems** describe two parts that are considered separately or in combination: Robin and Carly have 8 pencils all together; Carly has 3 pencils; how many does Robin have?
- **compare problems** contain two amounts to be compared for the difference between them: Robin has 5 pencils and Carly has 3 pencils; how many fewer pencils does Carly have than Robin? There are also two categories of compare problems, depending on whether the question is which has more or which has fewer.

Most children in the early elementary grades can use modelling to solve simple one-step problems, such as combine problems for which the answer is a whole number. For instance, they can represent the objects in the problem with manipulatives, tally marks, or their fingers and count to get the answer. As they get better at problem solving, they replace such cumbersome strategies with shorter, internalized ones that make the process more efficient. They also generalize their strategies so that they can apply them to new problems with a similar underlying mathematical structure (De Corte & Verschaffel, 2006). Proficient problem solving can be defined as the ability to represent a problem, decide on a solution procedure, and carry out that procedure. Predictably, children become proficient at addition and subtraction relatively quickly, while multiplication and division problems take longer to master (Anghileri, 2001; Clark & Kamii, 1996).

Children do not reach expert problem solving status without a few quirks, however. One interesting phenomenon that has been observed is the "suspension of sense making." Children seem to suffer a sort of logical oversight that prevents them from realizing when problems are false or absurd. For instance, when researchers gave students in Grades 1 and 2 the problem: "There are 26 sheep and 10 goats on a ship. How old is the captain?", the majority gave a numerical answer, most often 36 (Carpenter, Lindquist, Matthews, & Silver, 1983). Older elementary students are not immune to the effects of suspension of sense making. Students in Grade 8 were given the problem: "An army bus holds 36 soldiers. If 1,128 soldiers are being bussed to their training site, how many busses are needed?" The majority of students correctly divided 1,128 by 36, but less than a third used the remainder (12) to conclude that an extra bus was needed for these "left-over" individuals (Carpenter et al., 1983). While these older students were not as easily confused by absurd questions, they still did not apply their knowledge of the real world to their answers. They effectively suspended their sense-making abilities, resulting in very few "realistic" responses or comments on word problems such as this one. Research regularly identifies this effect and finds that it is strong and resistant to change (for a review, see Verschaffel, Greer, & De Corte, 2000).

Students whose problem-solving skills are still developing also tend to demonstrate a lack of strategic approaches, not to be confused with a lack of problem-solving strategies. When faced with a problem, children do not spontaneously respond by analyzing the problem, making a drawing of it, breaking it down into more manageable units, or other valuable strategies. That is, they rarely step back and consider the problem's context and elements before they attempt to solve it by applying a procedure. Even when given encouragement to take these steps, they do not significantly improve their performance (De Bock, Van Dooren, Janssens, & Verschaffel, 2002). This phenomenon is particularly common in students who have weak problem-solving skills (e.g., Hegarty, Mayer, & Monk, 1995). Unlike students with strong problem-solving skills, they tend to rely on superficial methods rather than on building a mental representation and carefully analyzing the problem.

Students' lack of strategic approaches is directly related to a lack of metacognitive activity during the problem-solving process, such as self-regulation, self-monitoring, and reflection. Good problem solvers self-regulate more often than poor

problem solvers do, and this is true both of younger and older children (Carr & Biddlecomb, 1998; Garofalo & Lester, 1985). In addition to conceptual understanding and computational fluency, students need to know how to approach problem solving strategically in order to succeed. They also must use self-regulation strategies while they work on the problem.

Proportional Reasoning

Proportionality is an important concept not just in mathematics and science, but also in everyday life, for instance to halve or double a recipe. A cake will not rise if we increase the other ingredients without increasing the baking powder by the same proportion. In mathematics, proportionality describes multiplicative relationships between rational quantities and is the basis for rational number operations, basic algebra, and problem solving in geometry (e.g., Fuson & Abrahamson, 2005; Saxe, Gearhart, & Seltzer, 1999; Sophian, Garyantes, & Chang, 1997). "The ability to reason proportionally develops in students [between] Grades 5 [and] 8. It is of such great importance that it merits whatever time and effort that must be expended to assure its careful development" (NCTM, 1989, p. 82). Some of the mathematical concepts relating to ratio and proportion include direct and indirect relations, linearity, rate of change, and scaling.

Proportional reasoning can be seen as analogical reasoning with quantities – both conceptual analogies and quantitative proportions require students to analyze the relations between relations (Boyer, Levine, & Huttenlocher, 2008). Although proportional reasoning is generally thought to develop in the later elementary grades, there has been disagreement in the research literature as to the age at which children are first able to use proportional reasoning successfully. It is a complex construct that varies according to number structures and context. Some studies have shown evidence for (somewhat modified) proportional reasoning in the early elementary years (e.g., Goswami, 1989; Sophian & Wood, 1997), but other studies support proportional reasoning only as a later achievement, after age eleven (e.g., Fujimura, 2001; Schwartz & Moore, 1998). Younger children can reason proportionately if the quantities involved are continuous rather than discrete (Spinillo & Bryant, 1999; Jeong, Levine, & Huttenlocher, 2007).

One type of strategy for dealing with problems involving proportional reasoning is multiplicative: the terms in the ratio are related multiplicatively. The first ratio is determined to be a:b, where b is a multiple of a. This relation is then extended to the second ratio. This is what we do when we double a recipe: everything is multiplied by two. This classical comparison of ratios underpins almost all the number-related concepts that are studied in school, including fractions, percentages, ratios, proportion, rates, similarity, trigonometry, and rates of change (Mitchelmore, White, & McMaster, 2007). Another strategy is called building-up and involves establishing the relationship of one ratio and extending that relationship to the second ratio by addition. This strategy is the dominant one observed in the majority of elementary students (Tourniaire & Pulos, 1985). Students apply both correct and incorrect strategies when attempting problems involving proportional reasoning.

CONCLUSION

Children begin their exploration of mathematics with a natural desire to discover the world around them. At a young age, they are curious, creative, and inquisitive risk-takers who use mathematics as a means to understand their surroundings.

Research has shown that young children require high quality, challenging, and accessible math education experiences in order to build a strong foundation for their future learning. What children are ready to learn in mathematics depends largely on their previous opportunities. In general, children learn by building on prior knowledge, and mathematics is particularly additive in nature. Concepts build on one another, so that early misunderstandings will impede further learning. A strong foundation in mathematics means a bright future: children's mathematics ability at the beginning of Kindergarten is a strong predictor of later academic success, even stronger than their early reading ability.

When children transition to school, they integrate their own intuitive understanding of mathematics with the new information from the more formal education system. As they progress through the elementary grades, children solidify the knowledge they have gained in the early years and deepen their conceptual understanding. They face new challenges, use new skills, and achieve understanding of new concepts.

Mathematics educators who have a good knowledge of their subject and who can put this knowledge into action in the classroom will be able to guide, support and augment children's developing understanding of mathematics. Educators' knowledge, behaviours, and attitudes related to mathematics are vital for student success. Educators have a great responsibility to provide children with a strong foundation in mathematics and thus enhance their chances for later academic success.

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3

Mathematical Development in the Elementary School Years

MATHEMATICAL DEVELOPMENT IN THE ELEMENTARY SCHOOL YEARS

What are the important aspects of cognitive development that influence how children acquire mathematical facts, knowledge, and procedures in the elementary school years? Children's cognitive development from ages six to twelve shows considerable growth. Six-year-olds are moving beyond intuitive and externalized interactions with the world. They start to form more complex mental representations of their own knowledge and that of others. This ability to consider another's perspective (referred to as having a "theory of mind") ties into increased working memory skills, both in terms of the amount of information children can retain, and in the availability of processing resources. The ways in which children encode and store information are still not exactly like those of adults, but they become more logical and more organized in how they make connections between different pieces of knowledge. This increase in memory capacity and memory organization develops gradually in the age range of six to twelve, but can also show periods of rapid change. Every child is an individual and so progresses on a unique trajectory – for teachers, this can make student learning in their classroom frustratingly variable. Nevertheless, as a group, children's developing ability to process new information, retain it, and thus to solve increasingly complex problems takes elementary school children from the rudiments of symbolic number system knowledge to basic algebra. Furthermore, as children progress through the stage described by Piaget as *concrete operational* (i.e., a period when children have a better understanding of mental operations and begin to think logically about concrete events; ages 7-12), to the beginnings of *formal operational* (i.e., a period when skills such as logical thought, deductive reasoning, and systematic planning emerge and the ability to think about abstract concepts develops; ages 12-adulthood), they become ready to question their existing knowledge and challenge their teachers and peers.

In the elementary years, children's mathematical development is closely tied to the experiences that they have in school. Initially, teachers and students must expend considerable effort to master the symbolic number system. Children must learn number vocabulary, knowledge of place value, rules for generating larger numbers, the conventions related to number labels and number symbols, and they must connect this symbolic knowledge to underlying representations of quantity. In Kindergarten, for example, a child who can label the symbol "37" with the verbal label "thirty-seven" is showing a reasonable level of knowledge for his or her age. By the end of Grade 1, children are expected to be producing labels for numbers in the hundreds and to understand how those symbols represent quantities. Symbolic knowledge allows children to compare and contrast large quantities in a precise way. Knowing that 134 is a smaller quantity than 341, for example, requires children to link quantities to symbols through conventional rules. Ideally, children start to connect quantities of numbers in the hundreds and thousands to other representations (such as a physical number line); by Grade 2 we expect that many children can accurately place a number such as 678 appropriately when shown a line with the endpoints labeled 0 and 1000.

Exactly when children acquire certain knowledge is of concern in relation to curriculum guidelines. It is important to realize that the goal is for all children to continue to learn and develop their knowledge. At times, children may be ahead of the curriculum and at other times they may be behind. Overall, the acquisition of knowledge will vary according to experiences, disposition, and even brain maturation.

Children are also expected to build up a repertoire of mathematical procedures in the first few years of school, starting with basic addition and subtraction in Grade 1 and moving to multiplication and division in Grades 3 and 4. Learning these operations involves more than memorizing "facts;" a conceptual understanding of additive composition helps children to understand subtraction and addition as complementary operations. Curriculum standards have helped to expand the scope of what children are exposed to in school. The National Council of Teachers of Mathematics (NCTM) has provided "focal points" to help teachers identify the central aspects of knowledge acquisition that help children to build a reasonably complete package of mathematical knowledge (http://www.nctm.org/standards/). In combination with an understanding of cognitive development in the elementary years, this information can help teachers to build up their own understanding of what constitutes successful mathematical learning in this age range.

Along with exciting increases in children's knowledge organization and capacity for learning come some other developments that can be frustrating for teachers. Because children in elementary school still view the world in concrete terms, they may over-apply rules that they have learned. This lack of flexibility, which may be generalized or may be specific to certain areas, might help children to acquire knowledge that is rule based, but may prevent them from seeing beyond the rule to the general principle. Teachers need to be aware that even when children exhibit excellent procedural skills in mathematics, they may nevertheless fail to understand *why* those procedures work, or may be unaware of when to apply certain rules. For example, some 9- and 10-year-olds may insist on solving a problem like $14 + 7 - 7$ by adding $14 + 7$ and then subtracting 7. Even if they know that $7 - 7 = 0$, they might not be confident in using the knowledge in the context of another problem. This tendency to apply superficial and rigid rules may, in part, account for another very typical mistake that children make. When presented with problems like $5 + 6 + 3 = 4 + ?$, they may simply add up all the numbers. By applying a typical procedure that usually works – that is, "add numbers to get the answer" – they are showing their lack of understanding of the meaning of the equal sign. Luckily, these kinds of misconceptions, once they are revealed, are relatively easy for teachers to anticipate and avoid. Understanding that children at this age tend to learn rules without necessarily extracting the bigger picture is useful in understanding mathematical development.

Although many children between six and twelve show both increasing capacity and a tendency to focus on rules and procedures, teachers also need to be aware that cognitive development and skill learning is variable and does not always seem to move in one direction. This means that a learner may revert to a less sophisticated strategy, even after showing evidence of using something more advanced. A researcher by the name of Robert Siegler has described development as a series of overlapping waves, where some skills are moving forward, some appear to be moving backwards, and in other cases, little learning appears to occur. As a group and over a reasonably long period of time, children's knowledge moves forward, but on a day-to-day basis, changes may not always be improvements! Experienced teachers will learn to recognize these apparent periods of decline in children's learning in particular areas and recognize that individual children may show different patterns of learning for different skills. This tendency for knowledge acquisition to act like waves in the sea is one reason why a variety of teaching approaches and learning activities will be beneficial for both teachers and students. Sometimes children just need more practice to make progress in learning a skill, but other times it will be more important to present the procedure or concept differently so that it gives children the opportunity to reconfigure their knowledge.

Knowing about general principles of learning can also be helpful for teachers. Many of the characteristics that describe children's learning apply more generally. Adults may appear to learn faster because they already have more knowledge that they can link to new information. Nevertheless, anyone learning a new skill shares similar challenges. Consider the situation of a North American who moves to the UK and wants to be knowledgeable about the game of cricket. She may have knowledge about similar games (such as baseball) that can form the basis of her mental representation for cricket. She will need to learn new vocabulary, new rules, and cope with inconsistencies where knowledge of baseball or other sports actually interferes with learning how the game of cricket works. If she perseveres, acquires the knowledge, goes and watches the games, or even participates, her understanding will increase. She may experience "ah-ha" moments where a particularly obscure referee's call suddenly makes sense. But her knowledge will accrue gradually, undergo reorganizations, sometimes desert her at critical moments, and most importantly, will require effort to acquire. Compare such a learning process with children's developing capacities of storage and processing efficiency, and it is clear that mathematical development from six to twelve will be variable across skill domains, across individual children, and as a function of the instructional milieu.

In summary, children's cognitive development from six through twelve allows them to acquire new knowledge and procedures, as well as the conceptual structures that support mathematical learning. Development will not always be smooth and uni-directional. Teachers can encourage student success in mathematics by becoming knowledgeable about children's cognitive development.

More detailed information about the various stages of development in relation to mathematical learning can be found in the following book:

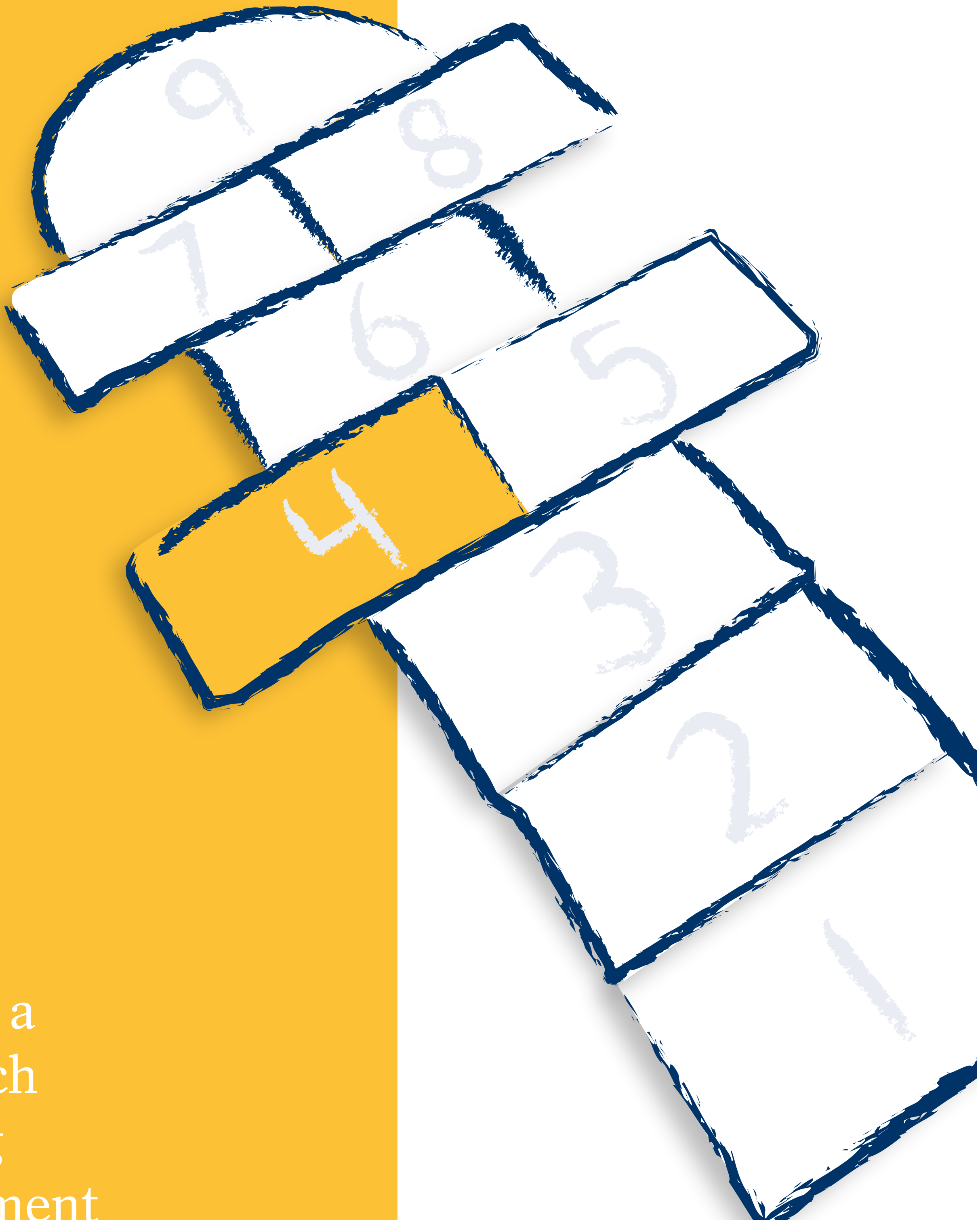
Siegler, R., & Alibali, M. W. (2004). *Children's thinking.* Upper Saddle River, NJ: Prentice Hall.

To learn more about children's development of a "theory of mind," refer to:

Astington, J. W. (1993). *The child's discovery of the mind*. In J. Bruner, M. Cole, & A. Karmiloff-Smith (Series Eds.). *The developing child*. Cambridge, MA: Harvard University Press.

For a link to learning in schools, consult:

Hattie, J. (2009). *Visible learning.* New York: Routledge.



4

Creating a Math-Rich Learning Environment

CREATING A MATH-RICH LEARNING ENVIRONMENT

A math-rich classroom is not one that requires fancy and expensive tools, but is instead one that is led by a teacher who is knowledgeable in the area of age-appropriate math concepts and who can incorporate them into daily activities (Sarama & Clements, 2005). Increasingly, the role of the teacher is viewed as a mentor who helps children connect their informal numeracy knowledge with their increasingly explicit knowledge of mathematics (Sarama & Clements, 2005). Teachers can also help children connect various math topics to each other, such as geometry and numbers (e.g., count the sides of a desk and then measure each side; Clements & Sarama, 2006).

Recent thinking in math education has shifted from an emphasis on the teaching and practicing of algorithms (e.g., memorizing arithmetic facts or completing pages of math problems) to focusing on reasoning and problem solving and their application to real world problems. There is also a focus on developing positive attitudes and feelings of self-efficacy toward mathematics (De Corte & Verschaffel, 2006). For example, De Corte and Verschaffel (2006) developed five guidelines which should be included in an effective mathematics learning environment:

1. Teach and support active, productive learning processes in all students maintaining a balance between individual exploration and systematic instruction/direction.
2. Promote the development of self-regulation so that students learn to be in charge of their own learning.
3. Create activities based on real-life situations applicable to future knowledge and skills and that enable the opportunity to collaborate with others.
4. Create occasions to gain widely applicable thinking and learning skills within the math content.
5. Foster a classroom atmosphere and culture that promotes students' open reflection and discussion on learning and problem-solving strategies.

Number Sense

It is important for children to master the connection between quantities and numbers in the first two or three years of school (Nunes, 2008). Quantities do not need to be a numerical value (e.g., you can compare two people's heights without knowing their exact numerical value or you can determine which pile of candies has more visually). Working with quantities helps children develop early mathematic abilities. Initially, this process can begin as children learn the relation between a set of objects and the numeric symbol (word or digit) that represents it (e.g., ** is 2). It is important that children understand that a numeric concept like "2" can apply to any set of two objects (Gelman & Gallistel, 1986). As a result, a math-rich environment at this early stage will provide children with multiple opportunities to make judgments about quantity (e.g., *Which of these boxes has more blocks? Which tower is taller?*) and to associate quantities with meaningful symbols (e.g., *Let's count the blocks and see which box has more. Let's measure which tower is taller using this ruler*). In this way, teachers promote the understanding that quantity is an important feature of sets, that number symbols represent specific quantities, and that quantity is a characteristic that can be applied to any set.

As children develop basic number sense, a math-rich classroom will present children with more complex understandings of quantity. For example, if we take one away from a group of blocks but add a different block we still have the same number as before. This concept can easily be demonstrated with any number of objects in the classroom (e.g., pinecones during fall theme or balls during a play session). By using everyday objects and demonstrating how quantity is affected by the addition and/or subtraction of additional elements, children learn that quantity is impacted in consistent and predictable ways by the addition or subtraction of objects. Teachers can also use these practical, real-world experiences to make links between changes in quantity and written Arabic statements of the change (e.g., 10–1+1=10). The relating of real-world terms and operations to Arabic symbols will be an important skill for solving word problems (Cummins, 1991).

Number Line

As discussed in the research summary, a child's development of a mental number line is essential to the ability to add and subtract as well as to estimate (Siegler & Booth, 2004). Forming a mental number line requires the ability to visualize abstract numbers, to order numbers by quantity, locate a given number along a line, and generate any portion of the number line that may be required for problem solving (Gervasoni, 2005). For example, understanding that 50 is a number beyond 40, but is half the way between 0 and 100, reflects that a child has the ability to understand the relation between numbers.

The classroom environment can be used to develop this skill. For example, count the first 100 days of school and create a number line banner around the room. Each day add another number to the line and create activities that use the number line. Also, refer to the calendar number line in a way that reflects the relations between milestones (i.e., *What day was half-way between the start of the year and a holiday?*). Have the children collect items (e.g., paperclips); find where their number of paperclips fits on the line, is it more or less than another child's? Refer to the number line to help children with their addition or subtraction.

Use the number line to present number patterns, such as counting by 2s, 5s, or 10s. The development of a child's mental number line is also important for estimation. Children will be better able to understand the relation between numbers if they are asked to approximate the relative position of various numbers on a number line (e.g., label a number line with only 1 and 100, then ask a child where on a number line they think 84 will fall, then have them check your visual number line to determine the actual position (Sigler & Opfer, 2003). The accuracy of a child's number line continues to develop throughout grade school (Siegler & Opfer, 2003).

Work with a number line can be adjusted for children with more advanced skills. For example, the visual number line can be changed to include decimals or can become a positive and

negative integer number line. This will allow children to answer questions and visualize the distance from zero, whether it be positive or negative (e.g., *Which one is further from zero, 54 or negative 54?*; de Hevia & Spelke, 2009).

Base-10

The base-10 system is foundational to later math skills; many children require instruction that focuses on this repeating decade structure (Geary, 2006). From Kindergarten to Grade 2, children need to understand that the word "ten" can represent one unit (1 ten) or ten individual units (10 ones) and that these representations are exchangeable (National Council of Teachers of Mathematics, 2004). Any number of concrete materials can be used to help children learn the concepts of place value. For example, you can have blocks or connecting cubes representing ones, tens, hundreds, or you could have children group popsicle sticks into groups of tens using elastic bands and then use the groups of 10 as well as individual sticks to count or represent various numbers depending on the activity. Children can then use these blocks to represent the base-10 composition of numbers (e.g., 36 is represented as 3 ten-units and 6 one-units and not just 36 separate units; Geary, 2006). Using popsicle groupings that are banded together can also be used to effectively represent the process of borrowing and carrying with demonstrations of multi-digit arithmetic (e.g., When subtracting 9 from 32, there are not enough "sticks" to take 9 from 2. So, we take one of the ten-units apart to create "12" in the ones category. Then take away 9, leaving 3 one-unit popsicle sticks and 2 ten-unit sets).

Addition/Subtraction/Fractions/Division/Multiplication

Simple arithmetic is likely one of the easiest techniques that teachers can incorporate into a math-rich classroom. Even the relatively complex concept of division can be easily demonstrated with sharing (e.g., *If there are 10 cookies and 5 children, how many cookies will each child get if we share them fairly?*). In preschool and early elementary school, sharing gives children a basic understanding of simple fractional relationships. For example, read the book "The Doorbell Rang" by P. Hutchins (1986), then ask the students questions about how the cookies can be shared as more and more children arrive at the house.

Children also benefit from learning that there are multiple strategies that can be used to solve a problem. A simple way to demonstrate different problem solving methods is to have children share how they arrived at an answer with the class; if several students share their strategy, then likely the class will be exposed to several solution methods (Caliandro, 2000). Another way to facilitate this type of learning is by having the children solve math problems in small groups and then have the groups present their work to the class (De Corte & Verschaffel, 2006).

Learning about money is a skill commonly acquired in the primary grades; teachers can have a jar full of change ready for children to use to solve real-world problems. Skills targeted with money include: coin recognition, values of coins, counting sets, equivalent collections, choosing coins to make a specific amount, and making change (Van de Walle, Karp, & Bay-Williams, 2010).

Geometry

A classroom can be rich in the variety of shapes available for students to manipulate. Desks may be square, the table a rectangle or circle, and a funnel cone shaped. The block set may contain a variety of triangles and other shapes to be manipulated, allowing the formation of new shapes (e.g., a parallelogram). Having these shapes easily visible allows students to begin making shape connections, measuring shapes, and estimating their size or volume depending on their grade level. In a math-rich classroom, it is important for teachers to demonstrate and model the use of the correct terms for geometric shapes (e.g., cube, sphere, rhombus, cone). Geometry is especially easy to incorporate with art, such as by asking the children to create pictures from a group of shapes or by labeling the shapes that children spontaneously create. Geometry can also be emphasized in explaining the effects of shading on three-dimensional objects.

Technology

The use of simple technology such as calculators and computers is a valuable component of elementary mathematics education. However, it is important to consider technology as an addition to learning "mental" or "by-hand" strategies, not as a replacement for them. Calculators may be particularly valuable, for example, for checking answers (Langrall, Mooney, Nisbet, & Jones, 2008). Students could be asked to use a calculator to check either their own or another student's work. It is also important to show children that errors can occur very easily, even with technology.

Computer programs that are designed to teach children math concepts using games have become very popular. Understandably, it is very compelling for both busy teachers and parents to use a self-directed math program that can provide children with a potentially engaging, fun, and consistent method to present math concepts while providing them with consistent practice (with instant feedback on accuracy). However, computer programs' efficacy for teaching purposes can be challenging. For example, Wilson, Maisterek, and Simmons (1996) noted that many computer games have amusing repercussions if children make errors. As a result, in their study, it seemed that children found it entertaining to deliberately make errors, which undermined the efficacy of the practice. Teachers must be aware that children may not be using a computer program effectively and that monitoring to ensure that a child stays "on task" is still required. As well, teachers and parents should remember that interaction with others cannot be replaced by even a well-designed computer program.

If children have reliable access to computers, teachers can use simple programs to teach data management and presentation methods to children. For example, even young children can learn to create a database for simple information and then use this information to create graphs. Unfortunately, the consistency of access to technology can pose a challenge to many classroom instructors. Taking stock of classroom resources and skills with computers can help to maximize the efficacy of technology's incorporation into the classroom.

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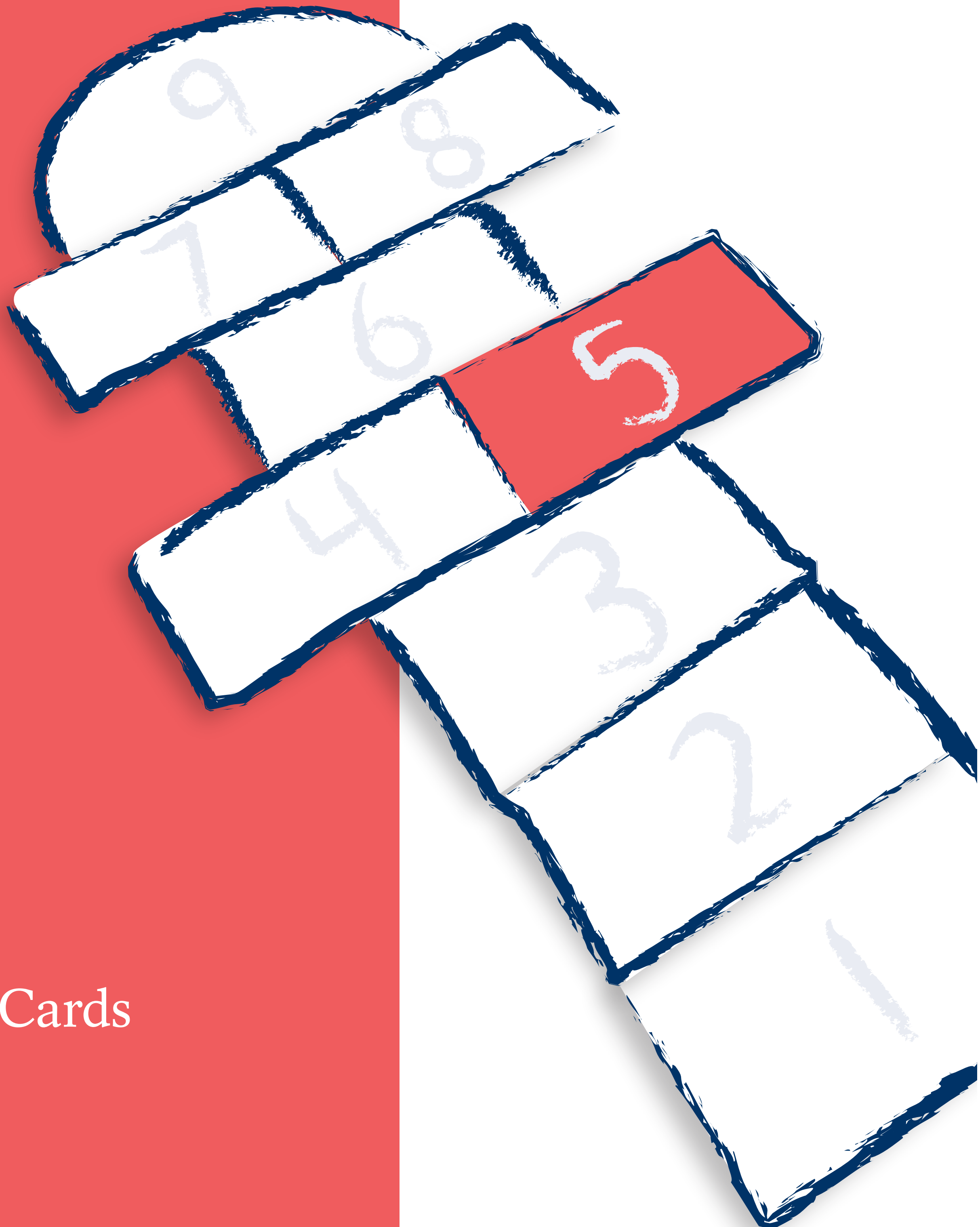
5

Activity Cards



ACTIVITY CARDS

The following activities are designed to target the mathematical concepts outlined in the research summary. These activities are only a small sample of the many creative ways that teachers can help their students enjoy learning mathematics.

The cards have been designed to remain in the resource or if you prefer, they can be cut on the perforated edge and used as individual cards.

Enjoy!

NAME OF ACTIVITY

Flyer Addition

I

MATH CONCEPT

Place value, operations with decimals, money

ESTIMATED GRADE LEVEL

Grade 6

MATERIALS

- weekly grocery flyers or leftover book orders
- notebook and writing materials
- money manipulatives

METHOD OF PRESENTATION

1. Ask students to make a list of items from the flyers that they wish to "purchase."
2. Have students add the total cost for each item.
3. Get students to demonstrate the coins needed to make their purchases using money manipulatives.

NAME OF ACTIVITY

Newspaper Scavenger Hunt

II

MATH CONCEPT

Number sense, problem solving

ESTIMATED GRADE LEVEL

Variable depending on information sought; can be used from Grade 1 to Grade 6

MATERIALS

newspapers, scissors, glue, cardstock, writing utensils

METHOD OF PRESENTATION

1. Ask students to identify where they see numbers in their everyday lives. Brainstorm a list with the class.
2. Have students identify different "real life numbers" that were brainstormed in a newspaper.
3. Cut these numbers out and glue them to a piece of cardstock.
4. Label these numbers with a description.

NAME OF ACTIVITY

Show Me with Dominoes

III

MATH CONCEPT

Number sense

ESTIMATED GRADE LEVEL

Variable depending on information sought; can be used from Grade 1 to Grade 6

MATERIALS

class set of dominoes

METHOD OF PRESENTATION

Working in small groups of three or four, have students identify the following in their domino set:

- more than or less than five dots
- the domino with the most/fewest dots
- even numbers on one half, odd numbers on the other half
- dominoes where both halves are the same number
- dominoes with the sum of six (or any other sum from 0 to 12 inclusive)
- the four dominoes that show the smallest (or largest) numbers
- the number of sides on a square (rectangle, triangle)
- the value of a Canadian coin
- a domino that, when both sides are added, demonstrate a number between 5 and 12
- a domino whose halves add to a number greater than six
- the number of days in a week

EXTENSION IDEAS

Give the students a set amount of money to spend. Determine exactly how many items that can purchase within their budget. Calculate the change given to students if a bill is used.

Flyer Addition

As a sample, have students find:

- a price
- an address
- a date
- a number that demonstrates a size
- a number that demonstrates a weight
- a phone number
- the date that the newspaper was published
- a number that demonstrates distance
- a temperature
- a score in a game
- an odd number
- an even number
- a number greater than 235
- a number less than 9879
- a number in the thousands
- a decimal

EXTENSION IDEAS

For each of the numbers identified, have the students write out each of the numbers in words. Next, have the students describe, in a sentence, where else they may find a similar number.

Newspaper Scavenger Hunt

EXTENSION IDEAS

Using the dominoes, have students add/subtract/multiple/divide the numbers that they see on each domino. For subtraction or division, what happens when the first number is smaller than the second number?

Show Me with Dominoes

NAME OF ACTIVITY

Calendar Cut Ups

IV

MATH CONCEPT
Number sense, problem solving, addition, subtraction, multiplication, division

ESTIMATED GRADE LEVEL
Grade 6

MATERIALS
- at least one calendar per student
- scissors
- notebook and writing utensils

METHOD OF PRESENTATION
1. Have each student cut apart the numbers from several calendar pages.
2. Arrange the numbers face down in the centre of each student's workspace.
3. Call out the following activities to the students, working in pairs:
- Choose three numbers. Multiply them.
- Choose five numbers. Find their average value.
- Pick four numbers. Add them. Divide the sum by six.
- Select three numbers. Multiply the larger two. Subtract the smaller value.
- Pick three numbers. Arrange them to make the largest possible number.
- Select six numbers. Make two larger numbers from them. Subtract the smaller from the larger number.
- Choose four numbers. Arrange them to make the smallest possible value. Read the number.

NAME OF ACTIVITY

Musical Numbers

V

MATH CONCEPT
Counting, one-to-one correspondence, addition, subtraction

ESTIMATED GRADE LEVEL
Grade 1

MATERIALS
- objects to count
- xylophone or musical instrument
- working space

METHOD OF PRESENTATION
1. Play each note a certain number of times, using a xylophone.
2. Ask students to put that many beans on their desk from a pile. Then ask how many beans there should be on each desk.
3. Run the stick across the xylophone to clear the board.
4. Repeat using different patterns of notes on the xylophone.

NAME OF ACTIVITY

Mystery Hands

VI

MATH CONCEPT
Counting, one-to-one correspondence, addition

ESTIMATED GRADE LEVEL
Grade 2

MATERIALS
- objects to count (i.e., beans)

METHOD OF PRESENTATION
1. Place students into small groups.
2. Person 'A' shakes given number of beans and breaks the total into two hands.
3. Person 'B' opens up one hand. The group verbalizes how many beans it sees.
4. On the magic word 'and' person 'A' opens up the second hand. The group again verbalizes how many beans it sees. (i.e., 2 and 4 equal 6)
5. Repeat with different quantities.

EXTENSION IDEAS

Make up and work on your own Calendar Cut Up activity. Write it in your journal to share with someone in your class. If time permits, trade activities with a friend.

Calendar Cut Ups

EXTENSION IDEAS

Ask the students to add their beans with their elbow partner's beans. How many beans are there in all? Repeat activity with subtraction.

Musical Numbers

EXTENSION IDEAS

Person A opens one hand and the group 'guesses' what must be in the other. The number to guess can be either one: ? + 4 = 6 or 2 + ? = 6. Repeat this activity with number cards to reinforce the visual number symbol and its equating 'bean' representation. Activity can be repeated with a variety of objects to increase complexity.

Mystery Hands

NAME OF ACTIVITY

What's in a Name?

VII

MATH CONCEPT

Counting, addition, money, decimals, number line

ESTIMATED GRADE LEVEL

Grade 4

Note: This activity can be adapted with different values for vowels and consonants to simplify addition and subtraction for younger grades.

MATERIALS

- notebook and writing utensils

METHOD OF PRESENTATION

1. Give students the following values.
2. If …..a = 1 ¢, b = 2 ¢, c = 3 ¢, d = 4¢, e = 5 ¢, f = 6¢, g = 7¢, h = 8¢, i = 9¢, j = 10¢, k = 11¢, l = 12¢, m = 13¢, n = 14¢, o = 15¢, p = 16¢, q = 17¢, r = 18¢, s = 19¢, t = 20¢, u = 21¢, v = 22¢, w = 23¢, x = 24¢, y = 25¢, z = 26¢

Ask students to determine the following:

How much is your first name worth?

Your last name?

What are you worth altogether?

Who is the most expensive in your class?

The least expensive? What does your teacher cost?

NAME OF ACTIVITY

Letters in our Names

VIII

MATH CONCEPT

Patterning and algebra, data management and probability, problem solving

ESTIMATED GRADE LEVEL

Grade 5 or 6

MATERIALS

- post-it notes
- chalkboard/ whiteboard

METHOD OF PRESENTATION

1. Using post-it notes, have students write their first name, one letter per note, and record the total number of letters in their first name on the extra note.
2. Rank the names from fewest to most letters, placing the extra post-it note to the left of the ranked names.
3. Once the names are posted, the numbers can be removed from the display and can be used to create a frequency table.
4. To interpret the data, have students record both displays in their notebooks and answer the following questions.
   - What is the mode (the most frequently occurring number)?
   - What is the median length of first names in our class?
   - Describe the overall shape of the data. Are there peaks, clusters, etc?
   - What is the mean length of first names?

NAME OF ACTIVITY

Fraction Kits

IX

MATH CONCEPT

Fractions

ESTIMATED GRADE LEVEL

Grades 4, 5 or 6

MATERIALS

- five 12 x 18 pieces of construction paper, each a different colour
- scissors
- rulers
- markers

METHOD OF PRESENTATION

1. Demonstrate and have students follow along with the same actions.
2. Cut the paper the long way into four 3" strips and trade so that each student has five strips of a different colour.
3. Fold the first strip into two equal pieces, cut and label the two halves as ½.
4. Fold the second strip into two equal pieces and then fold it again. Cut and label the pieces each ¼.
5. Fold the third strip into two equal pieces, fold again, and then again so that there are eight equal pieces. Cut and label the pieces each 1/8.
6. Fold the fourth strip into two equal pieces. Fold until there are sixteen equal pieces. Cut and label the pieces each 1/16.
7. The remaining strip should be labeled 1 to represent a whole.
8. Compare the different fraction pieces and how they relate to each other.

EXTENSION IDEAS
Use this activity in conjunction with the “Letters in our Names” activity. Alternatively, change the values of the letters to represent more difficult mathematical operations, involving decimals, negative numbers, etc.

What’s in a Name?

EXTENSION IDEAS
Use this activity in conjunction with the “What’s in a Name?” activity to determine which names would be most “expensive” to produce for a t-shirt printing company.

Letters in our Names

EXTENSION IDEAS
Use these fraction kits to play the “Cover Up” activity.

Fraction Kits

NAME OF ACTIVITY

Cover Up

X

MATH CONCEPT
Fractions, adding fractions, subtracting fractions, equivalent fractions

ESTIMATED GRADE LEVEL
Grade 6

MATERIALS
- fraction kit
- a cube labeled ½, ¼

METHOD OF PRESENTATION
1. Start with the whole strip.
2. Take turns rolling the die.
3. Put the fraction piece that matches the fraction on the cube on the whole strip.
4. First person to completely cover the whole strip is the winner. No overlapping is allowed.
5. Exact roll is needed to win.

NAME OF ACTIVITY

People Fractions

XI

MATH CONCEPT
Identifying fractions

ESTIMATED GRADE LEVEL
Grade 4

MATERIALS
n/a

METHOD OF PRESENTATION
1. Count the number of people in your class. Identify some characteristic within the group and ask:
   - How many people are there in the group?
   - How many of those people are wearing glasses?
   - How many of those people have long sleeves?
   - Etc.
2. Repeat saying the identified number out of the total group. You may want to write the determined fraction on the chalkboard to reinforce the written form.
   - There are six people in the group. Three of them are wearing glasses. This means that three of the six are wearing glasses, or three sixths.

NAME OF ACTIVITY

Baby Math

XII

MATH CONCEPT
Data management

ESTIMATED GRADE LEVEL
Grades 4 and 5

MATERIALS
- newspaper birth announcements
- graph paper
- pencil crayons
- rulers
- notebooks and writing utensils

METHOD OF PRESENTATION
1. Count the total number of babies born in your newspaper.
2. Record the number of baby boys and baby girls born on a frequency table.
3. Make a bar graph to demonstrate your results.
4. List the names of the babies alphabetically.

EXTENSION IDEAS

Variation: "Uncover." Play the game in the same way, only the whole strip is covered to begin the game. Each roll tells you what piece to remove. The exact roll is needed to end the game and empty the strip. Trading for equivalent fractions may be needed – be careful!

Cover Up

EXTENSION IDEAS

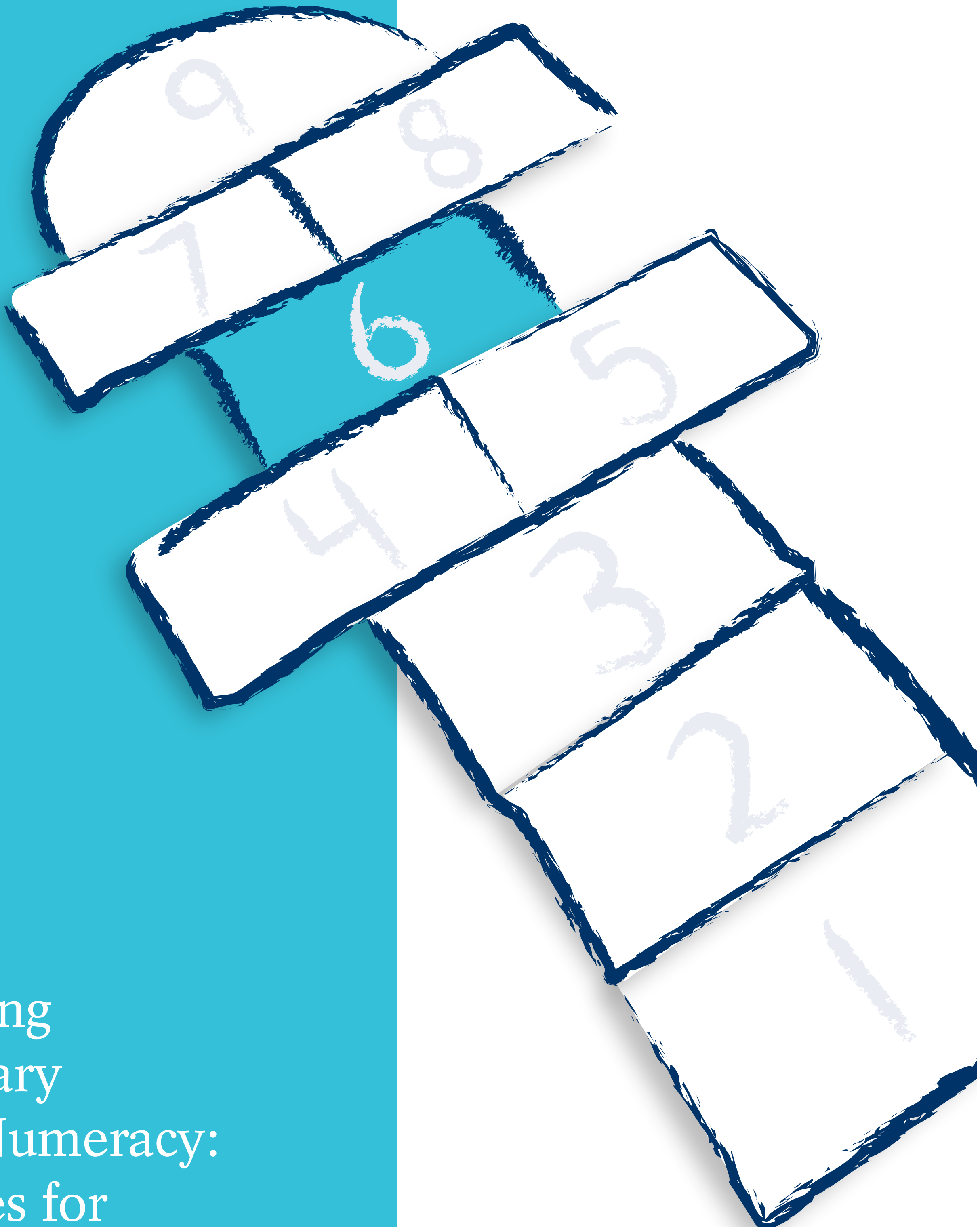
Make a class list of all of the fractions that the groups can determine about the students in your class. How can these fractions be represented in pictures, numbers and words?

People Fractions

EXTENSION IDEAS

Generate a class list of the months in which students were born. Create a frequency table and bar graph of this data. What is the mean month in which students were born?

Baby Math



6

Supporting Elementary School Numeracy: Resources for Mathematics Teachers

Recommended Online Resources

British Columbia Ministry of Education – Early Numeracy Project
Available at: http://www.bced.gov.bc.ca/numeracy/

The BC Ministry of Education funded a three-year research initiative called *The Early Numeracy Project* with a goal to enhance numeracy learning for children in Kindergarten to Grade 1, particularly those at-risk in the area of mathematics. Four tools for teachers have been developed from this research project including:

1. Assessing Early Numeracy
2. Supporting Early Numeracy
3. Whole Group Follow-up
4. Math for Families – Helping Your Child with Math at Home

Canadian Mathematical Society
Available at: http://www.cms.math.ca/

The Canadian Mathematical Society promotes and advances the discovery, learning and application of mathematics in Canada by fostering the community of mathematicians, promoting mathematical research, supporting education efforts at all levels and championing mathematics in the Canadian public. This website provides links to upcoming events, current research, journals, books and other important information.

Count Me In (Carleton University)
Available at: http://www.carleton.ca/cmi/index.htm

Count Me In is a research project funded by the Social Sciences and Humanities Research Council of Canada (SSHRC). Because one of the predictors of success in mathematics is the development of basic counting and number skills, this website has teacher and parent zones that provide teachers and parents with links to current numeracy resources and research to support the development of these skills. The kid zone section of this website provides online math activities for students from K-8.

Designed Instruction
Available at: http://www.designedinstruction.com/index.html

Designed Instruction is an education research and development firm from Texas that focuses on improving student learning through design and development of instructional products, education research and evaluation, and alignment of instructional products and programs with state and national U.S. education standards. This website has sections devoted to instructional resources including teacher activities, tips, and research for teachers and parents. The instructional resources are divided into PreKornerTM (early childhood education) and LearningLeadsTM (K-12) sections, both of which can be accessed from the main page.

Encyclopedia of Language and Literacy Development
Available at: http://literacyencyclopedia.ca/

This web-based resource developed by the Canadian Language and Literacy Research Network (CLLRNet) helps provide answers to questions about children's language, literacy and numeracy – answers that are based on relevant and up-to-date research presented in an easily accessible format. Teachers can all draw on the *Encyclopedia* for reliable, evidence-based information to support their daily practices. The Encyclopedia includes an extensive section on numeracy development, with contributions from leading numeracy researchers around the world. For sample entries, see chapters on *Acquisition of Mathematics in Primary School (ages 6 through 8) and Acquisition of Mathematics in Middle Childhood (ages 9 through 11).*

Illuminations
Available at: http://illuminations.nctm.org/

This online resource, developed by the National Council of Teachers of Mathematics (NCTM), contains over one hundred online activities and lessons arranged according to grade level and math strand. The site also provides links to hundreds of exemplary online resources.

JUMP Math
By: John Mighton
Available for purchase at: http://jumpmath.org/

JUMP Math is a numeracy program started in 1998 by mathematician, author, and award-winning playwright John Mighton. This program promotes the belief that all children can be led to think mathematically, and that with even a modest amount of attention every child will flourish. The program offers educators and parents complete and balanced materials as well as training to help them reach all students. Free sample worksheets are available to download from this website.

Manitoba Education
Available at: http://www.edu.gov.mb.ca/k12/cur/math/mathres.html

The Manitoba Education website provides mathematics learning resources for parents, teachers and students.

Mathematics Magazine for Grades 1-12
Available at: http://www.mathematicsmagazine.com/index.html

This magazine is an online publication for students and teachers. The magazine offers mathematics tests, listings of math tutors, mathematics applications and mathematics problems and exercises for Grades 1-12.

National Council of Teachers of Mathematics (NCTM)
Available at: http://www.nctm.org/

The *NCTM* is a public voice of mathematics education supporting teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development and research. This website includes information about the *NCTM*, curriculum focal points, lesson ideas, resources and current research. Links to the *NCTM*'s useful instructional resource series entitled *Navigation Series* are

available on this website. The *Navigation Series* was designed to provide teachers from pre-Kindergarten to Grade 8 with a wide variety of resources to facilitate mathematics program development and implementation.

Recommended Online Activities

Build It and See!
By: Digi-Block Inc.
Available for purchase at: http://www.digi-block.com/

The Digi-Block Program is a manipulative-based supplemental mathematics program that teaches base-10 number sense, place value and operations with whole numbers and decimals. *The Build It and See!* activity book presents 80 distinct challenges that reinforce critical learning concepts in number sense and operations. Available for Grades 1, 2, 3, and 4, the *Build It and See!* challenges support independent discovery by engaging students in problem solving and thinking about the base-ten number system. This resource is also available in Spanish.

The Centre for Education in Mathematics and Computing (CEMC) (University of Waterloo)
Available at: http://cemc.math.uwaterloo.ca/index.html

Founded in 1995, the *CEMC* has become Canada's largest and most recognized outreach organization for promoting and creating activities and materials in mathematics and computer science. This website provides links to research-based online activities including *Math Frog* that are suitable for children in Grades 4-12.

Esso Family Math
Available at: http://www.edu.uwo.ca/essofamilymath/index.asp

The *Esso Family Math Project* is a community-based initiative for families who want to help their children experience success in mathematics. It is a research-based program that was developed at the University of Western Ontario. Families learn to use everyday activities and materials to foster learning of mathematical concepts. These activities can be adapted for classroom use. Two books can be downloaded in PDF format, one for use with four- to six-year-olds and the other for seven- to ten-year-olds. They include lists of suggested books.

Family Math Fun!
By: Kate Nonesuch
Available at: http://www.nald.ca/library/learning/familymath/cover.htm

This ready-to-use manual of family numeracy activities can easily be adapted for the classroom. Activities include recipes, rhymes, games, and crafts. This manual was created as a result of collaboration between 30 parents in the Cowichan Valley and Kate Nonesuch, an instructor at Vancouver Island University. The manual can be downloaded in PDF format.

Figure This! Math Challenges for Families
Available at: http://www.figurethis.org/

This website was designed to help families enjoy mathematics outside school through a series of fun and engaging, high-quality challenges. A joint project by the National Council of Teachers of Mathematics (NCTM), the National Action Council for Minorities in Engineering (NACME), Widmeyer Communications, and the Learning First Alliance, this program provides challenging middle school mathematics activities and emphasizes the importance of high-quality math education for all students.

KidsNumbers.com
Available at: http://www.kidsnumbers.com/

This website offers free math resources designed by teachers, specifically for students and children of all ages. Students can practice all aspects of math, including addition, subtraction, multiplication and division, in a fun and pressure free way. Students can access many online activities, while teachers can use the site to create worksheets for their students.

KinderArt
Available at: http://www.kinderart.com/

This website offers some free ideas for simple math-related activities and lesson plans for preschool educators and K-12 teachers.

Math Dance
Available at: http://www.mathdance.org/

This site focuses on the relationship between movement and learning math. Teachers will find classroom activities that develop awareness of space, laterality and sequencing.

National Library of Virtual Manipulatives (NLVM)
By: Utah State University
Available at: http://nlvm.usu.edu/

The *NLVM*, created by the National Science Foundation, is a library of uniquely interactive, web-based virtual manipulatives or concept tutorials, mostly in the form of Java applets, for mathematics instruction (K-12 emphasis). It is designed for K-12 teachers who wish to enrich mathematics instruction with technology. The library is organized according to content area (number and operations, algebra, geometry, measurement and data analysis and probability) and grade level (Pre-K-2, 3-5, 6-8, 9-12).

PBS Parents – Early Math
Available at: http://www.pbs.org/parents/earlymath/

This website, from the U.S. Public Broadcasting System, is aimed at parents, but includes many activity ideas that can easily be adapted to the classroom. Suggestions for creative and fun activities are grouped by age, including pre-K to K and Grades 1 to 2. The site also includes simple online games and a list of math-related books for children.

Recommended Books and Print Resources

Canadian Berch, D. B., & Mazzocco, M. M. M. (Eds.). (2007). *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities.* Baltimore: Paul H. Brooks Publishing.

Based on the most current research available, this highly informative book gives readers the foundation they need to advance research, teaching strategies, and policies that identify struggling students and to begin developing appropriate practices that really help these students improve their math skills.

Burns, M. (2007). About teaching mathematics: *A K-8 resource* (3rd Ed.). Sausalito, CA: Math Solutions Publications.

This resource contains more than 240 classroom-tested lessons, and is essential in helping teachers: 1) to build student understanding and skills; and 2) to understand how children best learn math. In this third edition, Marilyn Burns has completely revised the first section to reflect what she has learned over the years from her classroom experience with students and her professional development experience with teachers. This section has also been expanded to address the following important topics: teaching math vocabulary, incorporating writing into math instruction, linking assessment and instruction, and using children's literature to teach key math concepts. In an entirely new section, the author addresses a wide range of questions she has received over the years from elementary and middle school teachers regarding classroom management and instructional issues.

Calkins, T. (2007). *Power of ten manual: Visual strategies for learning adding, subtracting and place value.* Victoria, BC: Power of Ten Educational Consulting.

This manual is a parent/teacher resource that provides a step-by-step program for teaching adding and subtraction using games and cards. This manual includes two 20-card decks, assessment sheets as well as many activity ideas based on non-traditional methods for adding, subtracting and place value. More information on this manual is available at http://poweroften.ca/index.php?option=com_content&view=article&id=45&Itemid=108

Canadian Language and Literacy Research Network (CLLRNet) & Canadian Child Care Federation (CCCF). (2009). *Foundations for numeracy: An evidence-based toolkit for early learning practitioners.* London, ON: CLLRNet.

This resource kit is designed for early learning practitioners. It includes a summary of research on the development of mathematics skills, along with practical suggestions/tools to help children succeed in mathematics in early learning and child care settings.

Colgan, L. (2003). *Elementary mathematics in Canada: Research summary and classroom implications.* Toronto, ON: Pearson Education, Canada Inc.

This resource examines the status of mathematics education in Canada and offers some practical suggestions on how to make a difference. It summarizes key research findings of best practice, suggests implications for the classroom and for professional development, and helps educators with the task of defining and shaping implementation plans.

Fennel, F., Bamberger, H. J., Rowan, T. E., Sammoms, K. B., & Suarez, A. R. (2000). *Series: Connect to NCTM Standards 2000: Making the standards work at grade (K-8).* Chicago, IL: Creative Publishing.

This series of books is designed to be read independently of the NCTM Principles and Standards for School Mathematics and to aid educators in preparing to teach in a manner consistent with the Principles and Standards. Each book is intended to model ways to develop content understanding along with process skills and to provide teachers the opportunity to incorporate what they have learned into lessons of their own creation. Teachers at the specific grade level of each book are the intended audience.

Kilpatrick, J., Swafford, J., & Findell B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Adding It Up explores how students in pre-Kindergarten through Grade 8 learn mathematics and recommends ways that teaching, curricula, and teacher education should change to improve mathematics learning during these critical years. The committee identifies five interdependent components of mathematical proficiency and describes how students develop this proficiency. With examples and illustrations, the book presents a portrait of mathematics learning: 1) research findings on what children know about numbers by the time they arrive in pre-K and the implications for mathematics instruction, and 2) details on the processes by which students acquire mathematical proficiency with whole numbers, rational numbers, and integers, as well as beginning algebra, geometry, measurement, and probability and statistics. The committee discusses what is known from research about teaching for mathematics proficiency, focusing on the interactions between teachers and students around educational materials and how teachers develop proficiency in teaching mathematics.

Krulik, S., & Rudnick, J. A. *Roads to reasoning: Developing thinking skills through problem solving grades 1-8 series.* Whitby, ON: McGraw-Hill Ryerson.

Roads to Reasoning falls into the categories of problem solving, standards and assessment, skill practice, and concept development. Each book contains six sections, each focusing on a different aspect of the problem solving process. The sections begin with a teaching problem followed by ten practice problems. As students practice their problem solving skills, their ability to reason will increase. The lessons in *Roads to Reasoning*

can be used as openers or warm-ups, as lessons of the day, as small group or partner activities, as individual assignments or even as homework.

Lester, F. K., & Charles, R. I. (Eds.). (2003). T*eaching mathematics through problem solving: pre-Kindergarten-Grade 6.* Reston, VA: National Council of Teachers of Mathematics.

This book presents several authors' research perspectives on teaching math through problem solving, including the benefits of teaching through problem solving, and designing and selecting problem solving tasks.

Mantyka, S. (2007). *The math plague: How to survive school mathematics.* St. John's, NL: MayT Consulting Corporation.

This book, written by Dr. Sherry Mantyka, Director of the Mathematics Learning Centre at Memorial University of Newfoundland, discusses the math difficulties that students face throughout school. The book is divided into 38 small sections, each containing a confirmed principle for the effective learning of mathematics; it is applicable to students of all ages as well as parents and teachers who are frustrated with the current state of mathematics.

Mighton, J. (2003). *The myth of ability: Nurturing the mathematical talent in every child.* Toronto, ON: Wilshire Publications.

John Mighton, the founder of the *JUMP (Junior Undiscovered Math Prodigies)* Math program describes his method of teaching the basics of mathematics in tiny, carefully-structured chunks that any student could understand. This book also highlights how all children are capable of learning, understanding and mastering mathematics but too often lose confidence in a culture where it is accepted that many will never fully understand the subject.

Mighton, J. (2008). *The end of ignorance: Multiplying our human potential.* Toronto, ON: Vintage Canada.

A follow-up to *The Myth of Ability*, John Mighton conceives a philosophy of education where all children have the potential to be successful, not only in mathematics but in every subject. He stresses that failure is a result of disenchantment in a subject and that self-confidence and perceived ability play key roles in academic success.

Schiro, M. S. (2009). *Mega-fun math games and puzzles for the elementary grades: 125 ready-to-use activities that teach math facts, concepts, and thinking skills.* San Francisco, CA: Jossey-Bass.

With this great collection of over 125 easy-to-use games, puzzles, and activities, teachers and parents can help kids comprehend fundamental math concepts, including addition, subtraction, multiplication, division, place value, fractions, and more. This resource is divided into two sections: the games section is designed specifically to help elementary students learn, remember and practice basic arithmetic facts, skills, and concepts by participating in enjoyable activities; the puzzle section focuses on developing algebra, geometry, measurement, probability, and data analysis. The games and puzzles are based on research on how children learn and understand mathematics. More information on this resource is available at http://www.wiley.com/WileyCDA/WileyTitle/productCd-047034475X.html

Silva, J. A. (2004). *Teaching inclusive mathematics to special learners, K-6.* Thousand Oaks, CA: Corwin Press.

This book is designed to open windows of understanding, which will help teachers to better recognize and compensate for the difficulties that special education students may encounter when learning mathematics. The author discusses general characteristics of students with learning disabilities as they relate to planning mathematics instruction as well as some instructional techniques and suggested modifications for facilitating their learning. Many of these techniques and strategies are designed to address specific difficulties or weaknesses; however, these techniques can be useful for all students, particularly those who may not be diagnosed with a learning disability, but have difficulty with math.

Small, M. (2008). *Making math meaningful to Canadian students, K-8.* Toronto, ON: Nelson.

Written for a Canadian audience, this book will start teachers on their way to a successful career in teaching mathematics by providing them with insight into how to make mathematics make sense to students and to capture their interest. Author Marian Small combines her wealth of research and practical experience to make this a thorough, yet very accessible text for students. This text is uniquely Canadian with samples from Canadian student texts and attention to Canadian curricula. It is an excellent reference for teachers who often have not had specialist training in mathematics, yet are expected to teach a more sophisticated curriculum to a diverse student population.

Small, M. (2009). *Good questions: Great ways to differentiate mathematics instruction.* New York: Teachers College Press.

This book was written to help K-8 teachers differentiate math instruction with less difficulty and greater success. This book: underscores the rationale for differentiating math instruction; describes two universal, easy-to-implement strategies designed to overcome the problems that teachers encounter; offers almost 300 questions and tasks that teachers and coaches can adopt immediately, adapt, or use as models to create their own; includes Teaching Tips sidebars and an organizing template at the end of each chapter to help readers build new tasks and open questions; and describes how to create a more inclusive classroom learning community with mathematical talk that engages participants from all levels.

Sowder, J., & Schappell, B. (Eds.). (2002). *Lessons learned from research.* Reston, VA: NCTM.

This book brings mathematics education research to K–12 teachers in an easy-to-use, readable form. It features 29 research articles from the U.S. Journal for Research in Mathematics Education, rewritten specifically for teachers. The chapters are collected into four broad sections: research related to: teaching; learning; curriculum; and assessment. The content is useful on many levels, from inspiration for classroom activities and assessment to explorations of the research conclusions and implications.

Sullivan, P., & Lilburn, P. (2002). *Good questions for math teaching: Why ask them and what to ask (K-6).* Sausalito, CA: Math Solutions Publications.

The purpose of this book is to provide elementary teachers with knowledge of good, open-ended mathematical questioning, defined by the authors as requiring "a student to think more deeply and to give a response that involves more than recalling a fact or reproducing a skill" (p. 1). The authors describe what makes good questions, how to create good questions, and give some practical ideas for using good questions in the classroom. The questions posed in this book cover concepts such as number, measurement, geometry, and data management and are organized according to age group. The intended audience is teachers of mathematics in grades K-6.

Van de Walle, J. A., & Folk, S. (2005). *Elementary and middle school mathematics: Teaching developmentally (Canadian edition).* Toronto: Pearson.

This book provides an unparalleled depth of ideas and discussion to help pre-service teachers develop a real understanding of the mathematics they will teach. John Van de Walle is one of the foremost experts on how children learn mathematics. The text reflects the NCTM Principles and Standards and the benefits of constructivist-or student-centered-mathematics instruction. Moreover, it is structured for maximum flexibility, offering 24 brief, compartmentalized chapters that may be mixed and matched to fit any course or teaching approach.

Van de Walle, J.A., & Lovin, L. H. (2006). *Teaching student-centered mathematics grades K-3, 3-5, 5-8*. Boston, MA: Pearson Education.

This series covers the best information available on how children learn mathematics and provides a one-stop resource of simple, problem-based activities designed to engage students in the mathematics that are important for them. The authors demonstrate how teachers can teach math in a student-centered, problem-based classroom, and help teachers to understand why this method is so successful in helping students to understand mathematics. The readable, user-friendly, completely integrated instructional strategies found throughout the series create a must-have three volume reference set.

Books for Children

The following list is to be used as a starting point for teachers when they are faced with the challenge of trying to select appropriate books to be used in mathematics instruction. This is not an exhaustive list and the grades are not exclusive. Teachers are encouraged to use a variety of books in the classroom for mathematics instruction.

Kindergarten to Grade 3

Adams, P. (2002). *Ten beads tall.* Wiltshire, UK: Child's Play International.

Bellfontaine, K. (2008). *Canada 1, 2, 3*. Toronto: Kids Can Press.

Brookes, D. (1990). *Passing the peace: A counting book for kids.* Manotick, ON: Penumbra Press.

Burns, M. (1994). *The greedy triangle*. Markham, ON: Scholastic.

Burns, M. (1990). *The $1.00 word riddle book.* Math Solutions Publications.

Emberley, E. (2001). *The wing on a flea: A book about shapes.* Little, Brown.

Kusugak, M. (1996). *My Arctic 1, 2, 3.* Toronto: Annick Press Ltd.

McFarlane, S. (2002). *A pod of orcas: A seaside counting book.* Toronto: Fitzhenry & Whiteside.

Organ, B. (2004). *My Newfoundland and Labrador counting book.* St. John's, NL: Creative Book Publishing.

Taylor, C. (2005). *Out on the prairie: A Canadian counting book.* Markham, ON: Scholastic.

Thornhill, J. (1990). *The wildlife 1, 2, 3: A nature counting book.* New York: Simon & Schuster.

Grades 3-6

Enzenberger, H. M. (1997). *The number devil: A mathematical adventure.* Markham, ON: Fitzhenry & Whiteside.

Neuschwander, C. (1997). *Sir cumference and the first round table*. Watertown, MA: Charlesbridge Publishing.

Neuschwander, C. (1999). *Sir cumference and the dragon of pi.* Watertown, MA: Charlesbridge Publishing.

Neuschwander, C. (2001). *Sir cumference and the great knight of angleland.* Watertown, MA: Charlesbridge Publishing.

Nolan, H., & Walker, T. (1995). *How much, how many, how far, how heavy, how long, how tall is 1000?* Toronto, ON: Kids Can Press Ltd.

Scieszka, J. (1995). *Math curse.* New York: The Penguin Group.

Tompert, A. (1990). *Grandfather tang's story.* New York: Crown Publishers.

Wells, R. E. (1993). *Is a blue whale the biggest thing there is?* Park Ridge, IL: Albert Whitman & Company.

Story Books

In addition to these books mentioned above, teachers are encouraged to use storybooks that involve comparing sizes and arranging items in order, for example, *Goldilocks and the Three Bears or The Three Billy Goats Gruff.*

Recommended Journals

PEER-REVIEWED JOURNALS

Canadian Journal of Mathematics (CJM)
Available at: http://journals.cms.math.ca/CJM

The *CJM* is a highly respected and strictly refereed journal published by the Canadian Mathematical Society to disseminate the timely and significant mathematical research vital to the worldwide scientific community. This internationally renowned journal maintains a vigorous publication schedule and is an independent, peer-reviewed journal devoted to publishing original works of high standard.

Canadian Mathematical Bulletin (CMB)
Available at: http://journals.cms.math.ca/CMB/

Appearing quarterly, the *CMB* is a highly respected and strictly refereed journal published by the Canadian Mathematical Society to disseminate the timely and significant mathematical research vital to the worldwide scientific community. This internationally renowned journal maintains a vigorous publication schedule and is an independent peer-reviewed journal devoted to publishing original works of high standard. The CMB publishes self-contained papers no longer than 15 typed pages. Readers include university and industry researchers who reflect a wide variety of fields of interest.

PROFESSIONAL JOURNALS

American Educator
Available at: http://www.aft.org/pubs-reports/american_educator/

American Educator, the professional journal of the American Federation of Teachers, is a quarterly magazine published for classroom teachers and other education professionals from preschool through university. Recent articles have focused on such topics as reducing the achievement gap between poor and affluent students; heading off student discipline problems; teaching an appreciation and understanding of democracy; the benefits of a common coherent curriculum; and other issues affecting children and education here and abroad.

For the Learning of Mathematics
Available at: http://flm.educ.ualberta.ca

This journal aims to stimulate reflection on mathematics education at all levels, and promote study of its practices and its theories: to generate productive discussion; to encourage enquiry and research; to promote criticism and evaluation of ideas and procedures current in the field. It is intended for the mathematics educator who is aware that the learning and teaching of mathematics are complex enterprises about which much remains to be revealed and understood.

Journal for Research in Mathematics Education (JRME)
Available at: http://www.nctm.org/publications/jrme.aspx

The *JRME*, an official journal of the National Council of Teachers of Mathematics, is devoted to the interests of teachers of mathematics and mathematics education at all levels-preschool through adult. *JRME* is a forum for disciplined inquiry into the teaching and learning of mathematics. The editors encourage the submission of a variety of manuscripts: reports of research, including experiments, case studies, surveys, philosophical studies, and historical studies; articles about research, including literature reviews and theoretical analyses; brief reports of research; critiques of articles and books; and brief commentaries on issues pertaining to research.

Mathematics Teaching in the Middle School (MTMS)
Available at: http://www.nctm.org/publications/mtms.aspx

MTMS is an official journal of the National Council of Teachers of Mathematics and is intended as a resource for middle school students, teachers, and teacher educators. The focus of the journal is on intuitive, exploratory investigations that use informal reasoning to help students develop a strong conceptual basis that leads to greater mathematical abstraction. The journal's articles have won numerous awards, including honors from the Society of National Association Publications.

Micromath
Available at: http://www.atm.org.uk/journal/micromath.html

Micromath is a journal of the Association of Mathematics Teachers that focuses on integrating technology and mathematics instruction.

Teaching Children Mathematics (TCM)
Available at: http://www.nctm.org/publications/tcm.aspx

TCM is an official journal of the National Council of Teachers of Mathematics and is intended as a resource for elementary school students, teachers, and teacher educators. The focus of the journal is on intuitive, exploratory investigations that use informal reasoning to help students develop a strong conceptual basis that leads to greater mathematical abstraction. The journal's articles have won numerous awards, including honors from the Society of National Association Publications.

7

Glossary of Terms Related to Elementary School Years Numeracy

A

Abstraction: One of the principles of counting: any group of objects can be counted, regardless of individual item type. For example, one orange, two pencils and three blocks can be counted (from 1 to 6) to find the total number of items.

Algebra: A branch of mathematics in which mathematical statements are used to describe relationships between variables, using letters and other symbols to represent numbers and number quantities.

Algorithm: In mathematics, an algorithm is a set of precise step-by-step instructions for how to arrive at an answer to a given problem; a formal procedure that is usually explicitly taught.

Associative Property: In addition and multiplication, the order in which three numbers are added or multiplied does not affect the sum or product. This is not true for subtraction and division. For example, (1 + 2) + 3 is the same as 1 + (2 + 3). (See also: Commutative Property.)

Automaticity: The quick, easy, and effortless (that is, the "automatic") retrieval of facts or procedures from long-term memory.

B

Base-10 System: The number system most commonly used in North America, based on grouping in tens. Ten is the base, and each place to the left is 10 times greater. For example, 100 = 10 times greater than 10. Each place value to the right of base-10 is one tenth of it (1/10); for example, 1 is 1/10, or one tenth, of 10.

Biologically Primary Knowledge: Inherent types of cognition, such as language and early quantitative competencies; usually emerge with little to no formal instruction, across all cultures.

Biologically Secondary Knowledge: Skills that build on biologically primary abilities and are cultural inventions (e.g., base-10 arithmetic).

C

Cardinality: One of the principles of counting and initial "how to count" rules: the value of the last number word used while counting indicates the quantity of items in the set. For example, counting "1, 2, 3" means that there are three items in the set.

Cardinal Numbers: The counting numbers (1, 2, 3, 4, 5, 6…) used to measure the size, or cardinality, of a set.

Change Problems: A type of word problem that contains some event that changes the value of a quantity. For example, "Robin has 5 pencils and Carly gives him 3 more. How many does Robin have now?"

Combine Problems: A type of word problem that describes two parts that are considered separately or in combination. For example, "Robin and Carly have 8 pencils all together. Carly has 3 pencils. How many does Robin have?"

Combining Units Strategy: A strategy used to solve arithmetic problems wherein the 100s, 10s, and units are dealt with separately. For example, solving 37 + 38 by 30 + 30 then 7 + 8.

Commutative Property: In addition and multiplication, the order in which two numbers are added or multiplied does not affect the sum or product. For example, the sum of 4 + 3 is the same as the sum of 3 + 4. (See also: Associative Property.)

Compare Problems: A type of word problem that contains two amounts to be compared for the difference between them. For example, "Robin has 5 pencils and Carly has 3 pencils. How many fewer pencils does Carly have than Robin?"

Compensating Strategy: A strategy used to solve arithmetic problems wherein the numbers are adjusted to simplify the arithmetic. For example, solving 37 + 38 as (35 + 35) + 2 + 3 = 75.

Conceptual Knowledge: Knowledge of why and how a mathematical procedure works, as well as general mathematical knowledge and understanding. For example, knowing that when counting, the last number stated represents the quantity of items in the set.

Concrete Operational Stage: One of Piaget's cognitive stages; a period when children have a better understanding of mental operations and begin to think logically about concrete events; ages 7-12.

Counting On: The ability to start at any number in the number sequence and continue counting from that number onward. (See also: Number-after Skill.)

Counting Skills: The ability to recite numbers in order. Children may be able to recite the number words in the correct order without understanding the underlying meaning.

D

Data: The information used as the basis of calculation.

Digit: The symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. For example, 4093 has four digits: 4, 0, 9, and 3.

E

Episodic Buffer: One of the three systems in the model of working memory; this memory storage system can integrate information across domains to form visual, spatial, and verbal information with time sequencing.

Estimation: An approximation of the exact value of an operation.

Extrinsic Motivation: The effort to learn made in the hope of some type of external reward such as good grades, a teacher or parent's praise, a sticker, etc.

F

Factual Knowledge: Knowledge of information that can be learned through memorization and repetition (e.g., 2 + 2 = 4), as well as memory of specific events and information.

Formal Operational Stage: One of Piaget's cognitive stages; a period in which skills such as logical thought, deductive reasoning, and systematic planning emerge and the ability to think about abstract concepts develops; ages 12-adulthood.

Fraction: Any part of a whole, number, or group. For example, ¼ or ½.

G, H, I

Geometry: A branch of mathematics that involves the study of shapes and configurations (e.g., straight lines, circles, etc).

Heuristic Methods: The systematic strategies that one uses for problem solving. (See also: Algorithm.)

Intrinsic Motivation: The desire to learn for the sheer enjoyment, challenge, pleasure, or interest of the activity.

L, M

Language-based Phonetic Buffer: One of the three systems in the model of working memory, this system is also known as the Phonological Loop. It temporarily stores the phonological, or auditory, information of language.

Manipulatives: Objects that children can handle (manipulate) to understand and work out simple arithmetic problems; for example, beans, buttons, or blocks. Children build their understanding of math with concrete objects before they move on to abstract number concepts.

Mastery-Oriented Goals: Students with mastery-oriented goals seek to master that which they are learning, focusing on their own achievement and attributing their success to effort. These students tend to challenge themselves and persist in the face of difficulty.

Math Anxiety: An emotional reaction, ranging from mild apprehension to fear or dread, in academic and everyday situations that deal with numbers.

Mental Number Line: A mental representation of numbers and relative magnitudes; requires the ability to visualize and abstract number, to order numbers by quantity, to locate a given number along an imaginary line, and to generate any portion of the number line that may be required for problem solving.

Metacognitive Knowledge: Also known as Metaknowledge; the knowledge of how, when, and why to use specific strategies or resources; what an individual knows about his or her own thinking.

Metacognitive Regulation: How one's knowledge is used to regulate and control one's own thinking.

N

Number-After Skill: The ability to state the next number in the counting sequence when one starts at any number. For instance, knowing that five is the next number after four without needing to count up from one. (See also: Counting On.)

Number Line: A horizontal line on which numbers are written in order from left to right. Once children have understood ordinality, they can look at the line and see that numbers farther to the right represent larger quantities than those on the left. As children learn about counting and about the concept of number itself, they start to generate a mental picture of the number line.

Number Sense: The understanding of number and operations; encompasses three subcomponents: 1) knowing about and using numbers; 2) knowing about and using operations; and 3) knowing about and using numbers and operations in computational settings.

Numeracy: A broad term that includes knowledge of number, arithmetic, procedures, problem solving, and measurement.

Numerosity: An approximate sense of number that babies as well as non-human animals (e.g., rats, lions, primates) have.

O

One-to-One Correspondence: One of the principles of counting and initial "how to count" rules: one, and only one, word can be assigned to each counted object. For example, an item in a set that has been assigned "3" cannot also be assigned "5".

Order-Irrelevance: One of the principles of counting: items in a set can be counted in any sequence and still reach the same total. For example, counting from right to left, left to right, or in no particular sequence at all will result in the same total number of items.

Ordinality: At its most basic level, the concept of *more* and *less*; develops to an understanding that higher numbers are associated with more items, and lower numbers with fewer items.

Ordinal Number: The number that refers to place or position (e.g., 1st, 2nd, 3rd).

P

Pattern: Any repeated design or recurring sequence. For example, the sequence 1, 2, 3, 1, 2, 3 or the flowers on wallpaper are both patterns.

Performance-Oriented Goals: Students with performance-oriented goals are focused outward, comparing their performance and learning to that of others; they tend to attribute success to ability, avoid challenging themselves, and give up when dealing with a difficult problem.

Place Value: The value of a digit based on its position in a number. For example, in the number '12' the digit '1' is in the 'tens' position and the digit '2' is in the 'ones' position, which indicates that 1 ten + 2 ones = 12).

Procedural Knowledge: Knowledge about how to complete an activity or task, including the motor sequences and skills needed. For example, knowing how to solve the problem 2 + 3 by continuing to counting on from 3 to get the sum – "4, 5."

Probability: The relative frequency with which an event will occur; expressed by the ratio of the number of actual occurrences to the total number of possible occurrences.

Proportionality: Refers to the multiplicative relationships between rational quantities; the basis for rational number operations, basic algebra, and problem solving in geometry.

S

Self-Efficacy: The set of beliefs one holds about one's own ability to succeed at difficult tasks.

Self-Regulation: A combination of motivation and cognitive processing; includes goal setting, planning, self-monitoring, evaluation, learning adjustments, and strategy choice.

Sequence: An ordered set of objects, numbers, shapes, etc. that are arranged according to a rule. For example, arranging dolls in order based on height, tallest to shortest.

Sequential Strategy: A strategy used to solve arithmetic problems wherein the value of the second number is counted up or down from the first number. For example, 37 + 38 is solved by 37 + 30 = 67, then 67 + 8 = 75.

Set: A collection of items that are grouped together; members of a set are called elements.

Stable Order: One of the principles of counting and initial "how to count" rules: the order in which number words are used to count objects is always the same. For example, counting in the order of "1, 2, 3" is correct, but "1, 2, 4" is incorrect.

T, V

Transfer: Also known as Learning Transfer; the ability to apply the skills and concepts used to solve one type of problem to another type of problem; learning can be applied beyond problems studied to both similar problems (Near Transfer) and to dissimilar problems (Far Transfer).

Visuospatial Sketch Pad: One of the three systems in the model of working memory; this system enables short term storage and manipulation of visual or spatial information.

W

Working Memory: Attention-driven control of information represented in the brain in one of three content-specific systems: the language-based phonetic buffer, the visuospatial sketch pad, and the episodic buffer.